

Linear Algebra

Hoffman & Kunze

2nd edition

Answers and Solutions to Problems and Exercises
Typos, comments and etc...

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Note

This is one of the ultimate classic textbooks in mathematics. It will probably be read for generations to come. Yet I cannot find a comprehensive set of solutions online. Nor can I find lists of typos. Since I'm going through this book in some detail I figured I'd commit some of my thoughts and solutions to the public domain so others may have an easier time than I have finding supporting materials for this book. Any book this classic should have such supporting materials readily available.

If you find any mistakes in these notes, please do let me know at one of these email addresses:

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Chapter 1: Linear Equations

Section 1.1: Fields

Page 3. Hoffman and Kunze comment that the term “characteristic zero” is “strange.” But the characteristic is the smallest n such that $n \cdot 1 = 0$. In a characteristic zero field the smallest such n is 0. This must be why they use the term “characteristic zero” and it doesn’t seem that strange.

Section 1.2: Systems of Linear Equations

Exercise 1: Verify that the set of complex numbers described in Example 4 is a subfield of \mathbb{C} .

Solution: Let $F = \{x + y\sqrt{2} \mid x, y \in \mathbb{Q}\}$. Then we must show six things:

1. 0 is in F
2. 1 is in F
3. If x and y are in F then so is $x + y$
4. If x is in F then so is $-x$
5. If x and y are in F then so is xy
6. If $x \neq 0$ is in F then so is x^{-1}

For 1, take $x = y = 0$. For 2, take $x = 1, y = 0$. For 3, suppose $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$. Then $x + y = (a + c) + (b + d)\sqrt{2} \in F$. For 4, suppose $x = a + b\sqrt{2}$. Then $-x = (-a) + (-b)\sqrt{2} \in F$. For 5, suppose $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$. Then $xy = (a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} \in F$. For 6, suppose $x = a + b\sqrt{2}$ where at least one of a or b is not zero. Let $n = a^2 + 2b^2$. Let $y = a/n + (-b/n)\sqrt{2} \in F$. Then $xy = \frac{1}{n}(a + b\sqrt{2})(a - b\sqrt{2}) = \frac{1}{n}(a^2 - 2b^2) = 1$. Thus $y = x^{-1}$ and $y \in F$.

Exercise 2: Let F be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{array}{l} x_1 - x_2 = 0 \quad 3x_1 + x_2 = 0 \\ 2x_1 + x_2 = 0 \quad x_1 + x_2 = 0 \end{array}$$

Solution: Yes the two systems are equivalent:

$$\begin{aligned} 3x_1 + x_2 &= \frac{1}{3}(x_1 - x_2) + \frac{4}{3}(2x_1 + x_2) \\ x_1 + x_2 &= \frac{-1}{3}(x_1 - x_2) + \frac{2}{3}(2x_1 + x_2) \\ x_1 - x_2 &= (3x_1 + x_2) - 2(x_1 + x_2) \\ 2x_1 + x_2 &= \frac{1}{2}(3x_1 + x_2) + \frac{1}{2}(x_1 + x_2) \end{aligned}$$

Exercise 3: Test the following systems of equations as in Exercise 2.

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 & x_1 - x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 & x_2 + x_3 &= 0 \\ \frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 &= 0 \end{aligned}$$

Solution: Yes the two systems are equivalent:

$$\begin{aligned} x_1 - x_3 &= \frac{-3}{4}(-x_1 + x_2 + 4x_3) + \frac{1}{4}(x_1 + 3x_2 + 8x_3) \\ x_2 + x_3 &= \frac{1}{4}(-x_1 + x_2 + 4x_3) + \frac{1}{4}(x_1 + 3x_2 + 8x_3) \end{aligned}$$

and

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= -(x_1 - x_3) + (x_2 + 3x_3) \\ x_1 + 3x_2 + 8x_3 &= (x_1 - x_3) + 3(x_2 + 3x_3) \\ \frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 &= \frac{1}{2}(x_1 - x_3) + (x_2 + 3x_3) \end{aligned}$$

Exercise 4: Test the following systems as in Exercise 2.

$$\begin{aligned} 2x_1 + (-1 + i)x_2 + x_4 &= 0 & (1 + \frac{i}{2})x_1 + 8x_2 - ix_3 - x_4 &= 0 \\ 3x_2 - 2ix_3 + 5x_4 &= 0 & \frac{2}{3}x_1 - \frac{1}{2}x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

Solution: These systems are not equivalent. Call the two equations in the first system E_1 and E_2 and the equations in the second system E'_1 and E'_2 . Then if $E'_2 = aE_1 + bE_2$ since E_2 does not have x_1 we must have $a = 1/3$. But then to get the coefficient of x_4 we'd need $7x_4 = \frac{1}{3}x_4 + 5bx_4$. That forces $b = \frac{4}{3}$. But if $a = \frac{1}{3}$ and $b = \frac{4}{3}$ then the coefficient of x_3 would have to be $-2i\frac{4}{3}$ which does not equal 1. Therefore the systems cannot be equivalent.

Exercise 5: Let F be a set which contains exactly two elements, 0 and 1. Define an addition and multiplication by the tables:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Solution: We must check the nine conditions on pages 1-2:

1. An operation is commutative if the table is symmetric across the diagonal that goes from the top left to the bottom right. This is true for the addition table so addition is commutative.
2. There are eight cases. But if $x = y = z = 0$ or $x = y = z = 1$ then it is obvious. So there are six non-trivial cases. If there's exactly one 1 and two 0's then both sides equal 1. If there are exactly two 1's and one 0 then both sides equal 0. So addition is associative.
3. By inspection of the addition table, the element called 0 indeed acts like a zero, it has no effect when added to another element.
4. $1 + 1 = 0$ so the additive inverse of 1 is 1. And $0 + 0 = 0$ so the additive inverse of 0 is 0. In other words $-1 = 1$ and

$-0 = 0$. So every element has an additive inverse.

5. As stated in 1, an operation is commutative if the table is symmetric across the diagonal that goes from the top left to the bottom right. This is true for the multiplication table so multiplication is commutative.

6. As with addition, there are eight cases. If $x = y = z = 1$ then it is obvious. Otherwise at least one of x, y or z must equal 0. In this case both $x(yz)$ and $(xy)z$ equal zero. Thus multiplication is associative.

7. By inspection of the multiplication table, the element called 1 indeed acts like a one, it has no effect when multiplied to another element.

8. There is only one non-zero element, 1. And $1 \cdot 1 = 1$. So 1 has a multiplicative inverse. In other words $1^{-1} = 1$.

9. There are eight cases. If $x = 0$ then clearly both sides equal zero. That takes care of four cases. If all three $x = y = z = 1$ then it is obvious. So we are down to three cases. If $x = 1$ and $y = z = 0$ then both sides are zero. So we're down to the two cases where $x = 1$ and one of y or z equals 1 and the other equals 0. In this case both sides equal 1. So $x(y + z) = (x + y)z$ in all eight cases.

Exercise 6: Prove that if two homogeneous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

Solution: Write the two systems as follows:

$$\begin{array}{rcl} a_{11}x + a_{12}y = 0 & & b_{11}x + b_{12}y = 0 \\ a_{21}x + a_{22}y = 0 & & b_{21}x + b_{22}y = 0 \\ \vdots & & \vdots \\ a_{m1}x + a_{m2}y = 0 & & b_{m1}x + b_{m2}y = 0 \end{array}$$

Each system consists of a set of lines through the origin $(0, 0)$ in the x - y plane. Thus the two systems have the same solutions if and only if they either both have $(0, 0)$ as their only solution or if both have a single line $ux + vy = 0$ as their common solution. In the latter case all equations are simply multiples of the same line, so clearly the two systems are equivalent. So assume that both systems have $(0, 0)$ as their only solution. Assume without loss of generality that the first two equations in the first system give different lines. Then

$$\frac{a_{11}}{a_{12}} \neq \frac{a_{21}}{a_{22}} \quad (1)$$

We need to show that there's a (u, v) which solves the following system:

$$\begin{array}{l} a_{11}u + a_{12}v = b_{i1} \\ a_{21}u + a_{22}v = b_{i2} \end{array}$$

Solving for u and v we get

$$\begin{aligned} u &= \frac{a_{22}b_{i1} - a_{12}b_{i2}}{a_{11}a_{22} - a_{12}a_{21}} \\ v &= \frac{a_{11}b_{i2} - a_{21}b_{i1}}{a_{11}a_{22} - a_{12}a_{21}} \end{aligned}$$

By (1) $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Thus both u and v are well defined. So we can write any equation in the second system as a combination of equations in the first. Analogously we can write any equation in the first system in terms of the second.

Exercise 7: Prove that each subfield of the field of complex numbers contains every rational number.

Solution: Every subfield of \mathbb{C} has characteristic zero since if F is a subfield then $1 \in F$ and $n \cdot 1 = 0$ in F implies $n \cdot 1 = 0$ in \mathbb{C} . But we know $n \cdot 1 = 0$ in \mathbb{C} implies $n = 0$. So $1, 2, 3, \dots$ are all distinct elements of F . And since F has additive inverses $-1, -2, -3, \dots$ are also in F . And since F is a field also $0 \in F$. Thus $\mathbb{Z} \subseteq F$. Now F has multiplicative inverses so $\pm \frac{1}{n} \in F$ for all natural numbers n . Now let $\frac{m}{n}$ be any element of \mathbb{Q} . Then we have shown that m and $\frac{1}{n}$ are in F . Thus their product $m \cdot \frac{1}{n}$ is in F . Thus $\frac{m}{n} \in F$. Thus we have shown all elements of \mathbb{Q} are in F .

Exercise 8: Prove that each field of characteristic zero contains a copy of the rational number field.

Solution: Call the additive and multiplicative identities of F 0_F and 1_F respectively. Define n_F to be the sum of n 1_F 's. So $n_F = 1_F + 1_F + \cdots + 1_F$ (n copies of 1_F). Define $-n_F$ to be the additive inverse of n_F . Since F has characteristic zero, if $n \neq m$ then $n_F \neq m_F$. For $m, n \in \mathbb{Z}, n \neq 0$, let $\frac{m}{n}_F = m_F \cdot n_F^{-1}$. Since F has characteristic zero, if $\frac{m}{n} \neq \frac{m'}{n'}$ then $\frac{m}{n}_F \neq \frac{m'}{n'}_F$. Therefore the map $\frac{m}{n} \mapsto \frac{m}{n}_F$ gives a one-to-one map from \mathbb{Q} to F . Call this map h . Then $h(0) = 0_F, h(1) = 1_F$ and in general $h(x + y) = h(x) + h(y)$ and $h(xy) = h(x)h(y)$. Thus we have found a subset of F that is in one-to-one correspondence to \mathbb{Q} and which has the same field structure as \mathbb{Q} .

Section 1.3: Matrices and Elementary Row Operations

Exercise 1: Find all solutions to the systems of equations

$$\begin{aligned}(1 - i)x_1 - ix_2 &= 0 \\ 2x_1 + (1 - i)x_2 &= 0.\end{aligned}$$

Solution: The matrix of coefficients is

$$\begin{bmatrix} 1 - i & -i \\ 2 & 1 - i \end{bmatrix}.$$

Row reducing

$$\rightarrow \begin{bmatrix} 2 & 1 - i \\ 1 - i & -i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 - i \\ 0 & 0 \end{bmatrix}$$

Thus $2x_1 + (1 - i)x_2 = 0$. Thus for any $x_2 \in \mathbb{C}$, $(\frac{1}{2}(i - 1)x_2, x_2)$ is a solution and these are all solutions.

Exercise 2: If

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}$$

find all solutions of $AX = 0$ by row-reducing A .

Solution:

$$\begin{aligned}\rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \\ 0 & 8 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1/7 \\ 0 & 8 & 2 \end{bmatrix} \\ \rightarrow \begin{bmatrix} 1 & 0 & 3/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 6/7 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & 3/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

Thus A is row-equivalent to the identity matrix. It follows that the only solution to the system is $(0, 0, 0)$.

Exercise 3: If

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

find all solutions of $AX = 2X$ and all solutions of $AX = 3X$. (The symbol cX denotes the matrix each entry of which is c times the corresponding entry of X .)

Solution: The system $AX = 2X$ is

$$\begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which is the same as

$$\begin{aligned}6x - 4y &= 2x \\4x - 2y &= 2y \\-x + 3z &= 2z\end{aligned}$$

which is equivalent to

$$\begin{aligned}4x - 4y &= 0 \\4x - 4y &= 0 \\-x + z &= 0\end{aligned}$$

The matrix of coefficients is

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the solutions are all elements of F^3 of the form (x, x, x) where $x \in F$.

The system $AX = 3X$ is

$$\begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which is the same as

$$\begin{aligned}6x - 4y &= 3x \\4x - 2y &= 3y \\-x + 3z &= 3z\end{aligned}$$

which is equivalent to

$$\begin{aligned}3x - 4y &= 0 \\x - 2y &= 0 \\-x &= 0\end{aligned}$$

The matrix of coefficients is

$$\begin{bmatrix} 3 & -4 & 0 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the solutions are all elements of F^3 of the form $(0, 0, z)$ where $z \in F$.

Exercise 4: Find a row-reduced matrix which is row-equivalent to

$$A = \begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} A &\rightarrow \begin{bmatrix} 1 & -2 & 1 \\ i & -(1+i) & 0 \\ 1 & 2i & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1+i & -i \\ 0 & 2+2i & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & \frac{1-i}{2} \\ 0 & 2+2i & -2 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & \frac{i-1}{2} \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & \frac{i-1}{2} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Exercise 5: Prove that the following two matrices are *not* row-equivalent:

$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix}.$$

Solution: Call the first matrix A and the second matrix B . The matrix A is row-equivalent to

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the matrix B is row-equivalent to

$$B' = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 3 page 7 $AX = 0$ and $A'X = 0$ have the same solutions. Similarly $BX = 0$ and $B'X = 0$ have the same solutions. Now if A and B are row-equivalent then A' and B' are row equivalent. Thus if A and B are row equivalent then $A'X = 0$ and $B'X = 0$ must have the same solutions. But $B'X = 0$ has infinitely many solutions and $A'X = 0$ has only the trivial solution $(0, 0, 0)$. Thus A and B cannot be row-equivalent.

Exercise 6: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 matrix with complex entries. Suppose that A is row-reduced and also that $a + b + c + d = 0$. Prove that there are exactly three such matrices.

Solution: Case $a \neq 0$: Then to be in row-reduced form it must be that $a = 1$ and $A = \begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$ which implies $c = 0$, so $A = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$. Suppose $d \neq 0$. Then to be in row-reduced form it must be that $d = 1$ and $b = 0$, so $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which implies $a + b + c + d \neq 0$. So it must be that $d = 0$, and then it follows that $b = -1$. So $a \neq 0 \Rightarrow A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Case $a = 0$: Then $A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$. If $b \neq 0$ then b must equal 1 and $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$ which forces $d = 0$. So $A = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}$ which implies (since $a + b + c + d = 0$) that $c = -1$. But c cannot be -1 in row-reduced form. So it must be that $b = 0$. So $A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$. If $c \neq 0$ then $c = 1$, $d = -1$ and $A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$. Otherwise $c = 0$ and $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus the three possibilities are:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Exercise 7: Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Solution: Write the matrix as

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

WOLOG we'll show how to exchange rows R_1 and R_2 . First add R_2 to R_1 :

$$\begin{bmatrix} R_1 + R_2 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

Next subtract row one from row two:

$$\begin{bmatrix} R_1 + R_2 \\ -R_1 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

Next add row two to row one again

$$\begin{bmatrix} R_2 \\ -R_1 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

Finally multiply row two by -1 :

$$\begin{bmatrix} R_2 \\ R_1 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}.$$

Exercise 8: Consider the system of equations $AX = 0$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 matrix over the field F . Prove the following:

- If every entry of A is 0, then every pair (x_1, x_2) is a solution of $AX = 0$.
- If $ad - bc \neq 0$, the system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$.
- If $ad - bc = 0$ and some entry of A is different from 0, then there is a solution (x_1^0, x_2^0) such that (x_1, x_2) is a solution if and only if there is a scalar y such that $x_1 = yx_1^0, x_2 = yx_2^0$.

Solution:

(a) In this case the system of equations is

$$\begin{aligned} 0 \cdot x_1 + 0 \cdot x_2 &= 0 \\ 0 \cdot x_1 + 0 \cdot x_2 &= 0 \end{aligned}$$

Clearly any (x_1, x_2) satisfies this system since $0 \cdot x = 0 \forall x \in F$.

(b) Let $(u, v) \in F^2$. Consider the system:

$$\begin{aligned} a \cdot x_1 + b \cdot x_2 &= u \\ c \cdot x_1 + d \cdot x_2 &= v \end{aligned}$$

If $ad - bc \neq 0$ then we can solve for x_1 and x_2 explicitly as

$$x_1 = \frac{du - bv}{ad - bc} \quad x_2 = \frac{av - cu}{ad - bc}.$$

Thus there's a unique solution for all (u, v) and in particular when $(u, v) = (0, 0)$.

(c) Assume WOLOG that $a \neq 0$. Then $ad - bc = 0 \Rightarrow d = \frac{cb}{a}$. Thus if we multiply the first equation by $\frac{c}{a}$ we get the second equation. Thus the two equations are redundant and we can just consider the first one $ax_1 + bx_2 = 0$. Then any solution is of the form $(-\frac{b}{a}y, y)$ for arbitrary $y \in F$. Thus letting $y = 1$ we get the solution $(-b/a, 1)$ and the arbitrary solution is of the form $y(-b/a, 1)$ as desired.

Section 1.4: Row-Reduced Echelon Matrices

Exercise 1: Find all solutions to the following system of equations by row-reducing the coefficient matrix:

$$\begin{aligned} \frac{1}{3}x_1 + 2x_2 - 6x_3 &= 0 \\ -4x_1 + 5x_3 &= 0 \\ -3x_1 + 6x_2 - 13x_3 &= 0 \\ -\frac{7}{3}x_1 + 2x_2 - \frac{8}{3}x_3 &= 0 \end{aligned}$$

Solution: The coefficient matrix is

$$\begin{bmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{bmatrix}$$

This reduces as follows:

$$\rightarrow \begin{bmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -7 & 6 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -67 \\ 0 & 48 & -134 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 6 & -18 \\ 0 & 1 & -67/24 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5/4 \\ 0 & 1 & -67/24 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$\begin{aligned} x - \frac{5}{4}z &= 0 \\ y - \frac{67}{24}z &= 0 \end{aligned}$$

Thus the general solution is $(\frac{5}{4}z, \frac{67}{24}z, z)$ for arbitrary $z \in F$.

Exercise 2: Find a row-reduced echelon matrix which is row-equivalent to

$$A = \begin{bmatrix} 1 & -i \\ 2 & 2 \\ u & 1+i \end{bmatrix}.$$

What are the solutions of $AX = 0$?

Solution: A row-reduces as follows:

$$\rightarrow \begin{bmatrix} 1 & -i \\ 1 & 1 \\ i & 1+i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -i \\ 0 & 1+i \\ 0 & i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -i \\ 0 & 1 \\ 0 & i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus the only solution to $AX = 0$ is $(0, 0)$.

Exercise 3: Describe explicitly all 2×2 row-reduced echelon matrices.

Solution:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Exercise 4: Consider the system of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 2x_2 &= 1 \\ x_1 - 3x_2 + 4x_3 &= 2 \end{aligned}$$

Does this system have a solution? If so, describe explicitly all solutions.

Solution: The augmented coefficient matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array} \right]$$

We row reduce it as follows:

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1/2 \\ 0 & -1 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{array} \right]$$

Thus the only solution is $(0, 0, 1/2)$.

Exercise 5: Give an example of a system of two linear equations in two unknowns which has no solutions.

Solution:

$$\begin{aligned} x + y &= 0 \\ x + y &= 1 \end{aligned}$$

Exercise 6: Show that the system

$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 + 7x_2 - 5x_3 - x_4 &= 3 \end{aligned}$$

has no solution.

Solution: The augmented coefficient matrix is as follows

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{array} \right]$$

This row reduces as follows:

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array} \right]$$

At this point there's no need to continue because the last row says $0x_1 + 0x_2 + 0x_3 + 0x_4 = -1$. But the left hand side of this equation is zero so this is impossible.

Exercise 7: Find all solutions of

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 &= -2 \\ 2x_1 - 4x_3 + 2x_4 + x_5 &= 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7 \end{aligned}$$

Solution: The augmented coefficient matrix is

$$\left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array} \right]$$

We row-reduce it as follows

$$\begin{aligned} \rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & -3 & -7 & 5 & 2 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array} \right] &\rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 4 & 4 & -4 & -1 & 7 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{array} \right] \\ \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

Thus

$$\begin{aligned} x_1 - 2x_3 + x_4 &= 1 \\ x_2 + x_3 - x_4 &= 2 \\ x_5 &= 1 \end{aligned}$$

Thus the general solution is given by $(1 + 2x_3 - x_4, 2 + x_4 - x_3, x_3, x_4, 1)$ for arbitrary $x_3, x_4 \in F$.

Exercise 8: Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

For which (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

Solution: The matrix A is row-reduced as follows:

$$\begin{aligned} &\rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \\ 0 & 8 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 7 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Thus for every (y_1, y_2, y_3) there is a (unique) solution.

Exercise 9: Let

$$\begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

For which (y_1, y_2, y_3, y_4) does the system of equations $AX = Y$ have a solution?

Solution: We row reduce as follows

$$\begin{aligned} &\left[\begin{array}{cccc|c} 3 & -6 & 2 & -1 & y_1 \\ -2 & 4 & 1 & 3 & y_2 \\ 0 & 0 & 1 & 1 & y_3 \\ 1 & -2 & 1 & 0 & y_4 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & y_4 \\ 3 & -6 & 2 & -1 & y_1 \\ -2 & 4 & 1 & 3 & y_2 \\ 0 & 0 & 1 & 1 & y_3 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & y_4 \\ 0 & 0 & -1 & -1 & y_1 - 3y_4 \\ 0 & 0 & 3 & 3 & y_2 + 2y_4 \\ 0 & 0 & 1 & 1 & y_3 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & y_4 \\ 0 & 0 & 0 & 0 & y_1 - 3y_4 + y_3 \\ 0 & 0 & 0 & 0 & y_2 + 2y_4 + 3y_3 \\ 0 & 0 & 1 & 1 & y_3 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & y_4 \\ 0 & 0 & 1 & 1 & y_3 \\ 0 & 0 & 0 & 0 & y_1 - 3y_4 + y_3 \\ 0 & 0 & 0 & 0 & y_2 + 2y_4 + 3y_3 \end{array} \right] \end{aligned}$$

Thus (y_1, y_2, y_3, y_4) must satisfy

$$\begin{aligned} y_1 + y_3 - 3y_4 &= 0 \\ y_2 + 3y_3 + 2y_4 &= 0 \end{aligned}$$

The matrix for this system is

$$\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

of which the general solution is $(-y_3 + 3y_4, -3y_3 - 2y_4, y_3, y_4)$ for arbitrary $y_3, y_4 \in F$. These are the only (y_1, y_2, y_3, y_4) for which the system $AX = Y$ has a solution.

Exercise 10: Suppose R and R' are 2×3 row-reduced echelon matrices and that the system $RX = 0$ and $R'X = 0$ have exactly the same solutions. Prove that $R = R'$.

Solution: There are seven possible 2×3 row-reduced echelon matrices:

$$R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

$$R_2 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix} \tag{3}$$

$$R_3 = \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4)$$

$$R_4 = \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix} \quad (5)$$

$$R_5 = \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix} \quad (6)$$

$$R_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (7)$$

$$R_7 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (8)$$

We must show that no two of these have exactly the same solutions. For the first one R_1 , any (x, y, z) is a solution and that's not the case for any of the other R_i 's. Consider next R_7 . In this case $z = 0$ and x and y can be anything. We can have $z \neq 0$ for R_2, R_3 and R_5 . So the only ones R_7 could share solutions with are R_3 or R_6 . But both of those have restrictions on x and/or y so the solutions cannot be the same. Also R_3 and R_6 cannot have the same solutions since R_6 forces $y = 0$ while R_3 does not.

Thus we have shown that if two R_i 's share the same solutions then they must be among R_2, R_4 , and R_5 .

The solutions for R_2 are $(-az, -bz, z)$, for z arbitrary. The solutions for R_4 are $(-a'y - b'z, y, z)$ for y, z arbitrary. Thus $(-b', 0, 1)$ is a solution for R_4 . Suppose this is also a solution for R_2 . Then $z = 1$ so it is of the form $(-a, -b, 1)$ and it must be that $(-b', 0, 1) = (-a, -b, 1)$. Comparing the second component implies $b = 0$. But if $b = 0$ then R_2 implies $y = 0$. But R_4 allows for arbitrary y . Thus R_2 and R_4 cannot share the same solutions.

The solutions for R_2 are $(-az, -bz, z)$, for z arbitrary. The solutions for R_5 are $(x, -a'z, z)$ for x, z arbitrary. Thus $(0, -a', 1)$ is a solution for R_5 . As before if this is a solution of R_2 then $a = 0$. But if $a = 0$ then R_2 forces $x = 0$ while in R_5 x can be arbitrary. Thus R_2 and R_5 cannot share the same solutions.

The solutions for R_4 are $(-ay - bz, y, z)$ for y, z arbitrary. The solutions for R_5 are $(x, -a'z, z)$ for x, z arbitrary. Thus setting $x = 1, z = 0$ gives $(1, 0, 0)$ is a solution for R_5 . But this cannot be a solution for R_4 since if $y = z = 0$ then first component must also be zero.

Thus we have shown that no two R_i and R_j have the same solutions unless $i = j$.

NOTE: This fact is actually true in genera not just for 2×3 ! (search for "1832109" on math.stackexchange).

Section 1.5: Matrix Multiplication

Page 18: Typo in the last paragraph where it says "the columns of B are the $1 \times n$ matrices ..." it should be $n \times 1$ not $1 \times n$.

Exercise 1: Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad C = [1 \quad -1].$$

Compute ABC and CAB .

Solution:

$$AB = \begin{bmatrix} 4 \\ 4 \end{bmatrix},$$

so

$$ABC = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \cdot [1 \ -1] = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}.$$

and

$$CBA = [1 \ -1] \cdot \begin{bmatrix} 4 \\ 4 \end{bmatrix} = [0].$$

Exercise 2: Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}.$$

Verify directly that $A(AB) = A^2B$.

Solution:

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix}. \end{aligned}$$

And

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} A^2B &= \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}. \end{aligned} \tag{9}$$

And

$$\begin{aligned} A(AB) &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}. \end{aligned} \tag{10}$$

Comparing (9) and (10) we see both calculations result in the same matrix.

Exercise 3: Find two different 2×2 matrices A such that $A^2 = 0$ but $A \neq 0$.

Solution:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

are two such matrices.

Exercise 4: For the matrix A of Exercise 2, find elementary matrices E_1, E_2, \dots, E_k such that

$$E_k \cdots E_2 E_1 A = I.$$

Solution:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}$$

$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & 0 \end{bmatrix}$$

$$E_3(E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 3 & 0 \end{bmatrix}$$

$$E_4(E_3 E_2 E_1 A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 3 & 0 \end{bmatrix}$$

$$E_5(E_4 E_3 E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 3/2 \end{bmatrix}$$

$$E_6(E_5 E_4 E_3 E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_7(E_6 E_5 E_4 E_3 E_2 E_1 A) = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_8(E_7 E_6 E_5 E_4 E_3 E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 5: Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}.$$

Is there a matrix C such that $CA = B$?

Solution: To find such a $C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ we must solve the equation

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}.$$

This gives a system of equations

$$\begin{aligned} a + 2b + c &= 3 \\ -a + 2b &= 1 \\ d + 2e + f &= -4 \\ -d + 2e &= 4 \end{aligned}$$

We row-reduce the augmented coefficient matrix

$$\begin{aligned} &\left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 3 \\ -1 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & -4 \\ 0 & 0 & 0 & -1 & 2 & 0 & 4 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cccccc|c} 1 & 0 & 1/2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1/4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1/2 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1/4 & 0 \end{array} \right]. \end{aligned}$$

Setting $c = f = 4$ gives the solution

$$C = \begin{bmatrix} -1 & 0 & 4 \\ -6 & -1 & 4 \end{bmatrix}.$$

Checking:

$$\begin{bmatrix} -1 & 0 & 4 \\ -6 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}.$$

Exercise 6: Let A be an $m \times n$ matrix and B an $n \times k$ matrix. Show that the columns of $C = AB$ are linear combinations of the columns of A . If $\alpha_1, \dots, \alpha_n$ are the columns of A and $\gamma_1, \dots, \gamma_k$ are the columns of C then

$$\gamma_j = \sum_{r=1}^n B_{rj} \alpha_r.$$

Solution: The ij -th entry of AB is $\sum_{r=1}^n A_{ir} B_{rj}$. Since the term B_{rj} is independent of i , we can view the sum independent of i as $\sum_{r=1}^n B_{rj} \alpha_r$ where α_r is the r -th column of A . I'm not sure what more to say, this is pretty immediately obvious from the definition of matrix multiplication.

Exercise 7: Let A and B be 2×2 matrices such that $AB = I$. Prove that $BA = I$.

Solution: Suppose

$$\begin{aligned} A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}. \\ AB &= \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}. \end{aligned}$$

Then $AB = I$ implies the following system in u, r, s, t has a solution

$$\begin{aligned} au + bs &= 1 \\ cu + ds &= 0 \\ ar + bt &= 0 \\ cr + dt &= 1 \end{aligned}$$

because (x, y, z, w) is one such solution. The augmented coefficient matrix of this system is

$$\left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ c & 0 & d & 0 & 0 \\ 0 & a & 0 & b & 0 \\ 0 & c & 0 & d & 1 \end{array} \right]. \quad (11)$$

As long as $ad - bc \neq 0$ this system row-reduces to the following row-reduced echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d/(ad - bc) \\ 0 & 1 & 0 & 0 & -b/(ad - bc) \\ 0 & 0 & 1 & 0 & -c/(ad - bc) \\ 0 & 0 & 0 & 1 & a/(ad - bc) \end{array} \right]$$

Thus we see that necessarily $x = d/(ad - bc)$, $y = -b/(ad - bc)$, $z = -c/(ad - bc)$, $w = a/(ad - bc)$. Thus

$$B = \begin{bmatrix} d/(ad - bc) & -b/(ad - bc) \\ -c/(ad - bc) & a/(ad - bc) \end{bmatrix}.$$

Now it's a simple matter to check that

$$\begin{bmatrix} d/(ad - bc) & -b/(ad - bc) \\ -c/(ad - bc) & a/(ad - bc) \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The loose end is that we assumed $ad - bc \neq 0$. To tie up this loose end we must show that if A is invertible then necessarily $ad - bc \neq 0$. Suppose that $ad - bc = 0$. We will show there is no solution to (11). If $a = b = c = d = 0$ then obviously A has no inverse. So suppose WOLOG that $a \neq 0$ (because by elementary row operations we can move any of the four elements to be the top left entry). Subtracting $\frac{c}{a}$ times the 3rd row from the 4th row of (11) gives

$$\left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ c & 0 & d & 0 & 0 \\ 0 & a & 0 & b & 0 \\ 0 & c - \frac{c}{a}a & 0 & d - \frac{c}{a}b & 1 \end{array} \right].$$

Now $c - \frac{c}{a}a = 0$ and since $ad - bc = 0$ also $d - \frac{c}{a}b = 0$. Thus we get

$$\left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ c & 0 & d & 0 & 0 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

and it follows that A is not invertible.

Exercise 8: Let

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

be a 2×2 matrix. We inquire when it is possible to find 2×2 matrices A and B such that $C = AB - BA$. Prove that such matrices can be found if and only if $C_{11} + C_{22} = 0$.

Solution: We want to know when we can solve for a, b, c, d, x, y, z, w such that

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

The right hand side is equal to

$$\begin{bmatrix} bz - cy & ay + bw - bx - dy \\ cx + dz - az - cw & cy - bz \end{bmatrix}$$

Thus the question is when can we choose a, b, c, d so that the following system has a solution for x, y, z, w

$$\begin{aligned} bz - cy &= c_{11} \\ ay + bw - bx - dy &= c_{12} \\ cx + dz - az - cw &= c_{21} \\ cy - bz &= c_{22} \end{aligned} \tag{12}$$

The augmented coefficient matrix for this system is

$$\left[\begin{array}{cccc|c} 0 & -c & b & 0 & c_{11} \\ -b & a-d & 0 & b & c_{12} \\ c & 0 & d-a & -c & c_{21} \\ 0 & c & -b & 0 & c_{22} \end{array} \right]$$

This matrix is row-equivalent to

$$\left[\begin{array}{cccc|c} 0 & -c & b & 0 & c_{11} \\ -b & a-d & 0 & b & c_{12} \\ c & 0 & d-a & -c & c_{21} \\ 0 & 0 & 0 & 0 & c_{11} + c_{22} \end{array} \right]$$

from which we see that necessarily $c_{11} + c_{22} = 0$.

Suppose conversely that $c_{11} + c_{22} = 0$. We want to show $\exists A, B$ such that $C = AB - BA$.

We first handle the case when $c_{11} = 0$. We know $c_{11} + c_{22} = 0$ so also $c_{22} = 0$. So C is in the form

$$\begin{bmatrix} 0 & c_{12} \\ c_{21} & 0 \end{bmatrix}.$$

In this case let

$$A = \begin{bmatrix} 0 & c_{12} \\ -c_{21} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} & AB - BA \\ &= \begin{bmatrix} 0 & c_{12} \\ -c_{21} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & c_{12} \\ -c_{21} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ c_{21} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -c_{12} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & c_{12} \\ c_{21} & 0 \end{bmatrix} \\ &= C. \end{aligned}$$

So we can assume going forward that $c_{11} \neq 0$. We want to show the system (12) can be solved. In other words we have to find a, b, c, d such that the system has a solution in x, y, z, w . If we assume $b \neq 0$ and $c \neq 0$ then this matrix row-reduces to the following row-reduced echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & (d-a)/c & -1 & \frac{d-a}{bc}c_{11} - \frac{c_{12}}{b} \\ 0 & 1 & -d/c & 0 & -c_{11}/c \\ 0 & 0 & 0 & 0 & -\frac{c_{11}}{b}(d-a) + c_{21} + \frac{c_{12}c}{b} \\ 0 & 0 & 0 & 0 & c_{11} + c_{22} \end{array} \right]$$

We see that necessarily

$$-\frac{c_{11}}{b}(d-a) + c_{21} + \frac{c_{12}c}{b} = 0.$$

Since $c_{11} \neq 0$, we can set $a = 0$, $b = c = 1$ and $d = \frac{c_{12} + c_{21}}{c_{11}}$. Then the system can be solved and we get a solution for any choice of z and w . Setting $z = w = 0$ we get $x = c_{21}$ and $y = c_{11}$.

Summarizing, if $c_{11} \neq 0$ then:

$$\begin{aligned} a &= 0 \\ b &= 1 \\ c &= 1 \\ d &= (c_{12} + c_{21})/c_{11} \\ x &= c_{21} \\ y &= c_{11} \\ z &= 0 \\ w &= 0 \end{aligned}$$

For example if $C = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$ then $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$. Checking:

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}.$$

Section 1.6: Invertible Matrices

Exercise 1: Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}.$$

Find a row-reduced echelon matrix R which is row-equivalent to A and an invertible 3×3 matrix P such that $R = PA$.

Solution: As in Exercise 4, Section 1.5, we row reduce and keep track of the elementary matrices involved. It takes nine steps to put A in row-reduced form resulting in the matrix

$$P = \begin{bmatrix} 3/8 & -1/4 & 3/8 \\ 1/4 & 0 & -1/4 \\ 1/8 & 1/4 & 1/8 \end{bmatrix}.$$

Exercise 2: Do Exercise 1, but with

$$A = \begin{bmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{bmatrix}.$$

Solution: Same story as Exercise 1, we get to the identity matrix in nine elementary steps. Multiplying those nine elementary matrices together gives

$$P = \begin{bmatrix} 1/3 & -\frac{29+3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{10} & \frac{1-3i}{10} \\ -i/3 & \frac{3+i}{15} & \frac{3+i}{5} \end{bmatrix}.$$

Exercise 3: For each of the two matrices

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

use elementary row operations to discover whether it is invertible, and to find the inverse in case it is.

Solution: For the first matrix we row-reduce the augmented matrix as follows:

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 6 & 4 & 1 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 0 & -11 & 4 & -2 & 1 & 0 \\ 0 & -11 & 4 & -3 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 0 & -11 & 4 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array} \right] \end{aligned}$$

At this point we see that the matrix is not invertible since we have obtained an entire row of zeros.

For the second matrix we row-reduce the augmented matrix as follows:

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 4 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -2 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 5 & -2 & -3 & 1 & 0 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 8 & -3 & 1 & -5 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3/8 & 1/8 & -5/8 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -3/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -3/8 & 1/8 & -5/8 \end{array} \right] \end{aligned}$$

Thus the inverse matrix is

$$\begin{bmatrix} 1 & 0 & 1 \\ -3/4 & 1/4 & -1/4 \\ 3/8 & 1/8 & -5/8 \end{bmatrix}$$

Exercise 4: Let

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}.$$

For which X does there exist a scalar c such that $AX = cX$?

Solution:

$$\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

implies

$$5x = cx \tag{13}$$

$$x + 5y = cy \tag{14}$$

$$y + 5z = cz \tag{15}$$

Now if $c \neq 5$ then (13) implies $x = 0$, and then (14) implies $y = 0$, and then (15) implies $z = 0$. So it is true for $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ with $c = 0$.

If $c = 5$ then (14) implies $x = 0$ and (15) implies $y = 0$. So if $c = 5$ any such vector must be of the form $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$ and indeed any such vector works with $c = 5$.

So the final answer is any vector of the form $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$.

Exercise 5: Discover whether

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

is invertible, and find A^{-1} if it exists.

Solution: We row-reduce the augmented matrix as follows:

$$\begin{aligned} & \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/4 \end{array} \right] \end{aligned}$$

Thus the A does have an inverse and

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}.$$

Exercise 6: Suppose A is a 2×1 matrix and that B is a 1×2 matrix. Prove that $C = AB$ is not invertible.

Solution: Write $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $B = [b_1 \ b_2]$. Then

$$AB = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}.$$

If any of a_1, a_2, b_1 or b_2 equals zero then AB has an entire row or an entire column of zeros. A matrix with an entire row or column of zeros is not invertible. Thus assume a_1, a_2, b_1 and b_2 are non-zero. Now if we add $-a_2/a_1$ of the first row to the second row we get

$$\begin{bmatrix} a_1 b_1 & a_1 b_2 \\ 0 & 0 \end{bmatrix}.$$

Thus AB is not row-equivalent to the identity. Thus by Theorem 12 page 23, AB is not invertible.

Exercise 7: Let A be an $n \times n$ (square) matrix. Prove the following two statements:

- (a) If A is invertible and $AB = 0$ for some $n \times n$ matrix B then $B = 0$.
- (b) If A is not invertible, then there exists an $n \times n$ matrix B such that $AB = 0$ but $B \neq 0$.

Solution:

(a) $0 = A^{-1}0 = A^{-1}(AB) = (A^{-1}A)B = IB = B$. Thus $B = 0$.

(b) By Theorem 13 (ii) since A is not invertible $AX = 0$ must have a non-trivial solution v . Let B be the matrix all of whose columns are equal to v . Then $B \neq 0$ but $AB = 0$.

Exercise 8: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Prove, using elementary row operations, that A is invertible if and only if $(ad - bc) \neq 0$.

Solution: Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}.$$

Then $AB = I$ implies the following system in u, r, s, t has a solution

$$\begin{aligned} au + bs &= 1 \\ cu + ds &= 0 \\ ar + bt &= 0 \\ cr + dt &= 1 \end{aligned}$$

because (x, y, z, w) is one such solution. The augmented coefficient matrix of this system is

$$\left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ c & 0 & d & 0 & 0 \\ 0 & a & 0 & b & 0 \\ 0 & c & 0 & d & 1 \end{array} \right]. \quad (16)$$

As long as $ad - bc \neq 0$ this system row-reduces to the following row-reduced echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d/(ad - bc) \\ 0 & 1 & 0 & 0 & -b/(ad - bc) \\ 0 & 0 & 1 & 0 & -c/(ad - bc) \\ 0 & 0 & 0 & 1 & a/(ad - bc) \end{array} \right]$$

Thus we see that $x = d/(ad - bc)$, $y = -b/(ad - bc)$, $z = -c/(ad - bc)$, $w = a/(ad - bc)$ and

$$A^{-1} = \begin{bmatrix} d/(ad - bc) & -b/(ad - bc) \\ -c/(ad - bc) & a/(ad - bc) \end{bmatrix}.$$

Now it's a simple matter to check that

$$\begin{bmatrix} d/(ad - bc) & -b/(ad - bc) \\ -c/(ad - bc) & a/(ad - bc) \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now suppose that $ad - bc = 0$. We will show there is no solution. If $a = b = c = d = 0$ then obviously A has no inverse. So suppose WOLOG that $a \neq 0$ (because by elementary row operations we can move any of the four elements to be the top left entry). Subtracting $\frac{c}{a}$ times the 3rd row from the 4th row of (16) gives

$$\left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ c & 0 & d & 0 & 0 \\ 0 & a & 0 & b & 0 \\ 0 & c - \frac{c}{a}a & 0 & d - \frac{c}{a}b & 1 \end{array} \right].$$

Now $c - \frac{c}{a}a = 0$ and since $ad - bc = 0$ also $d - \frac{c}{a}b = 0$. Thus we get

$$\left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ c & 0 & d & 0 & 0 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

and it follows that A is not invertible.

Exercise 9: An $n \times n$ matrix A is called **upper-triangular** if $a_{ij} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from zero.

Solution: Suppose that $a_{ii} \neq 0$ for all i . Then we can divide row i by a_{ii} to give a row-equivalent matrix which has all ones on the diagonal. Then by a sequence of elementary row operations we can turn all off diagonal elements into zeros. We can therefore row-reduce the matrix to be equivalent to the identity matrix. By Theorem 12 page 23, A is invertible.

Now suppose that some $a_{ii} = 0$. If *all* a_{ii} 's are zero then the last row of the matrix is all zeros. A matrix with a row of zeros cannot be row-equivalent to the identity so cannot be invertible. Thus we can assume there's at least one i such that $a_{ii} \neq 0$. Let i' be the largest such index, so that $a_{i'i'} = 0$ and $a_{ii} \neq 0$ for all $i > i'$. We can divide all rows with $i > i'$ by a_{ii} to give ones on the diagonal for those rows. We can then add multiples of those rows to row i' to turn row i' into an entire row of zeros.

Since again A is row-equivalent to a matrix with an entire row of zeros, it cannot be invertible.

Exercise 10: Prove the following generalization of Exercise 6. If A is an $m \times n$ matrix and B is an $n \times m$ matrix and $n < m$, then AB is not invertible.

Solution: There are n columns in A so the vector space generated by those columns has dimension no greater than n . All columns of AB are linear combinations of the columns of A . Thus the vector space generated by the columns of AB is contained in the vector space generated by the columns of A . Thus the column space of AB has dimension no greater than n . Thus the column space of the $m \times m$ matrix AB has dimension less or equal to n and $n < m$. Thus the columns of AB generate a space of dimension strictly less than m . Thus AB is not invertible.

Exercise 11: Let A be an $n \times m$ matrix. Show that by means of a finite number of elementary row and/or column operations one can pass from A to a matrix R which is both 'row-reduced echelon' and 'column-reduced echelon,' i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$ if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

Solution: First put A in row-reduced echelon form, R' . So \exists an invertible $m \times m$ matrix P such that $R' = PA$. Each row of R' is either all zeros or starts (on the left) with zeros, then has a one, then may have non-zero entries after the one. Suppose row i has a leading one in the j -th column. The j -th column has zeros in all other places except the i -th, so if we add a multiple of this column to another column then it only affects entries in the i -th row. Therefore a sequence of such operations can turn this row into a row of all zeros and a single one.

Let B be the $n \times n$ matrix such that $B_{rr} = 1$ and $B_{rs} = 0 \forall r \neq s$ except $B_{jk} \neq 0$. Then AB equals A with B_{jk} times column j added to column k . B is invertible since any such operation can be undone by another such operation. By a sequence of such operations we can turn all values after the leading one into zeros. Let Q be a product of all of the elementary matrices B involved in this transformation. Then PAQ is in row-reduced and column-reduced form.

Exercise 12: The result of Example 16 suggests that perhaps the matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

is invertible and A^{-1} has integer entries. Can you prove that?

Solution: This problem seems a bit hard for this book. There are a class of theorems like this, in particular these are called Hilbert Matrices and a proof is given in this article on arxiv by Christian Berg called *Fibonacci numbers and orthogonal polynomials* (<http://arxiv.org/pdf/math/0609283v2.pdf>). See Theorem 4.1. Also there might be a more elementary proof in this discussion on mathoverflow.net where two proofs are given:

<http://mathoverflow.net/questions/47561/deriving-inverse-of-hilbert-matrix>.

Also see <http://vigo.ime.unicamp.br/HilbertMatrix.pdf> where a general formula for the i, j entry of the inverse is given explicitly as

$$(-1)^{i+j}(i+j-1) \binom{n+i-1}{n-j} \binom{n+j-1}{n-i} \binom{i+j-1}{i-1}^2$$

Chapter 2: Vector Spaces

Section 2.1: Vector Spaces

Exercise 1: If F is a field, verify that F^n (as defined in Example 1) is a vector space over the field F .

Solution: Example 1 starts with any field and defines the objects, the addition rule and the scalar multiplication rule. We must show the set of n -tuples satisfies the eight properties required in the definition.

1) Addition is commutative. Let $\alpha = (x_1, \dots, x_n)$ and $\beta = (y_1, \dots, y_n)$ be two n -tuples. Then $\alpha + \beta = (x_1 + y_1, \dots, x_n + y_n)$. And since F is commutative this equals $(y_1 + x_1, \dots, y_n + x_n)$, which equals $\beta + \alpha$. Thus $\alpha + \beta = \beta + \alpha$.

2) Addition is associative. Let $\alpha = (x_1, \dots, x_n)$, $\beta = (y_1, \dots, y_n)$ and $\gamma = (z_1, \dots, z_n)$ be three n -tuples. Then $(\alpha + \beta) + \gamma = ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n)$. And since F is associative this equals $(x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n))$, which equals $\alpha + (\beta + \gamma)$.

3) We must show there is a unique vector 0 in V such that $\alpha + 0 = \alpha \forall \alpha \in V$. Consider $(0_F, \dots, 0_F)$ the vector of all 0 's of length n , where 0_F is the zero element of F . Then this vector satisfies the property that $(0_F, \dots, 0_F) + (x_1, \dots, x_n) = (0_F + x_1, \dots, 0_F + x_n) = (x_1, \dots, x_n)$ since $0_F + x = x \forall x \in F$. Thus $(0_F, \dots, 0_F) + \alpha = \alpha \forall \alpha \in V$. We must just show this vector is unique with respect to this property. Suppose $\beta = (x_1, \dots, x_n)$ also satisfies the property that $\beta + \alpha = \alpha$ for all $\alpha \in V$. Let $\alpha = (0_F, \dots, 0_F)$. Then $(x_1, \dots, x_n) = (x_1 + 0_F, \dots, x_n + 0_F) = (x_1, \dots, x_n) + (0_F, \dots, 0_F)$ and by definition of β this equals $(0_F, \dots, 0_F)$. Thus $(x_1, \dots, x_n) = (0_F, \dots, 0_F)$. Thus $\beta = \alpha$ and the zero element is unique.

4) We must show for each vector α there is a unique vector β such that $\alpha + \beta = 0$. Suppose $\alpha = (x_1, \dots, x_n)$. Let $\beta = (-x_1, \dots, -x_n)$. Then β has the required property $\alpha + \beta = 0$. We must show β is unique with respect to this property. Suppose also $\beta' = (x'_1, \dots, x'_n)$ also has this property. Then $\alpha + \beta = 0$ and $\alpha + \beta' = 0$. So $\beta = \beta + 0 = \beta + (\alpha + \beta') = (\beta + \alpha) + \beta' = 0 + \beta' = \beta'$.

5) Let 1_F be the multiplicative identity in F . Then $1_F \cdot (x_1, \dots, x_n) = (1 \cdot x_1, \dots, 1 \cdot x_n) = (x_1, \dots, x_n)$ since $1_F \cdot x = x \forall x \in F$. Thus $1_F \alpha = \alpha \forall \alpha \in V$.

6) Let $\alpha = (x_1, \dots, x_n)$. Then $(c_1 c_2) \alpha = ((c_1 c_2)x_1, \dots, (c_1 c_2)x_n)$ and since multiplication in F is associative this equals $(c_1(c_2 x_1), \dots, c_1(c_2 x_n)) = c_1(c_2 x_1, \dots, c_2 x_n) = c_1 \cdot (c_2 \alpha)$.

7) Let $\alpha = (x_1, \dots, x_n)$ and $\beta = (y_1, \dots, y_n)$. Then $c(\alpha + \beta) = c(x_1 + y_1, \dots, x_n + y_n) = (c(x_1 + y_1), \dots, c(x_n + y_n))$ and since multiplication is distributive over addition in F this equals $(cx_1 + cy_1, \dots, cx_n + cy_n)$. This then equals $(cx_1, \dots, cx_n) + (cy_1, \dots, cy_n) = c(x_1, \dots, x_n) + c(y_1, \dots, y_n) = c\alpha + c\beta$. Thus $c(\alpha + \beta) = c\alpha + c\beta$.

8) Let $\alpha = (x_1, \dots, x_n)$. Then $(c_1 + c_2)\alpha = ((c_1 + c_2)x_1, \dots, (c_1 + c_2)x_n)$ and since multiplication distributes over addition in F this equals $(c_1 x_1 + c_2 x_1, \dots, c_1 x_n + c_2 x_n) = (c_1 x_1, \dots, c_1 x_n) + (c_2 x_1, \dots, c_2 x_n) = c_1(x_1, \dots, x_n) + c_2(x_1, \dots, x_n) = c_1 \alpha + c_2 \alpha$. Thus $(c_1 + c_2)\alpha = c_1 \alpha + c_2 \alpha$.

Exercise 2: If V is a vector space over the field F , verify that

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4$$

for all vectors $\alpha_1, \alpha_2, \alpha_3$ and α_4 in V .

Solution: This follows associativity and commutativity properties of V :

$$\begin{aligned} & (\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) \\ &= (\alpha_2 + \alpha_1) + (\alpha_3 + \alpha_4) \\ &= \alpha_2 + [\alpha_1 + (\alpha_3 + \alpha_4)] \\ &= \alpha_2 + [(\alpha_1 + \alpha_3) + \alpha_4] \\ &= [\alpha_2 + (\alpha_1 + \alpha_3)] + \alpha_4 \\ &= [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4. \end{aligned}$$

Exercise 3: If C is the field of complex numbers, which vectors in \mathbb{C}^3 are linear combinations of $(1, 0, -1)$, $(0, 1, 1)$, and $(1, 1, 1)$?

Solution: If we make a matrix out of these three vectors

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

then if we row-reduce the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array} \right].$$

Therefore the matrix is invertible and $AX = Y$ has a solution $X = A^{-1}Y$ for any Y . Thus any vector $Y \in \mathbb{C}^3$ can be written as a linear combination of the three vectors. Not sure what the point of making the base field \mathbb{C} .

Exercise 4: Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$

$$c(x, y) = (cx, y).$$

Is V , with these operations, a vector space over the field of real numbers?

Solution: No it is not a vector space because $(0, 2) = (0, 1) + (0, 1) = 2(0, 1) = (2 \cdot 0, 1) = (0, 1)$. Thus we must have $(0, 2) = (0, 1)$ which implies $1 = 2$ which is a contradiction in the field of real numbers.

Exercise 5: On \mathbb{R}^n , define two operations

$$\alpha \oplus \beta = \alpha - \beta$$

$$c \cdot \alpha = -c\alpha.$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?

Solution:

1) \oplus is not commutative since $(0, \dots, 0) \oplus (1, \dots, 1) = (-1, \dots, -1)$ while $(1, \dots, 1) \oplus (0, \dots, 0) = (1, \dots, 1)$. And $(1, \dots, 1) \neq (-1, \dots, -1)$.

2) \oplus is not associative since $((1, \dots, 1) \oplus (1, \dots, 1)) \oplus (2, \dots, 2) = (0, \dots, 0) \oplus (2, \dots, 2) = (-2, \dots, -2)$ while $(1, \dots, 1) \oplus ((1, \dots, 1) \oplus (2, \dots, 2)) = (1, \dots, 1) \oplus (-1, \dots, -1) = (2, \dots, 2)$.

3) There does exist a right additive identity, i.e. a vector 0 that satisfies $\alpha + 0 = \alpha$ for all α . The vector $\beta = (0, \dots, 0)$ satisfies $\alpha + \beta = \alpha$ for all α . And if $\beta' = (b_1, \dots, b_n)$ also satisfies $(x_1, \dots, x_n) + \beta' = (x_1, \dots, x_n)$ then $x_i - b_i = x_i$ for all i and thus $b_i = 0$ for all i . Thus $\beta = (0, \dots, 0)$ is unique with respect to the property $\alpha + \beta = \alpha$ for all α .

4) There do exist right additive inverses. For the vector $\alpha = (x_1, \dots, x_n)$ clearly only α itself satisfies $\alpha \oplus \alpha = (0, \dots, 0)$.

5) The element 1 does not satisfy $1 \cdot \alpha = \alpha$ for any non-zero α since otherwise we would have $1 \cdot (x_1, \dots, x_n) = (-x_1, \dots, -x_n) = (x_1, \dots, x_n)$ only if $x_i = 0$ for all i .

6) The property $(c_1 c_2) \cdot \alpha = c_1 \cdot (c_2 \cdot \alpha)$ does not hold since $(c_1 c_2) \alpha = (-c_1 c_2) \alpha$ while $c_1 (c_2 \alpha) = c_1 (-c_2 \alpha) = (-c_1 (-c_2 \alpha)) = +c_1 c_2 \alpha$. Since $c_1 c_2 \neq -c_1 c_2$ for all c_1, c_2 they are not always equal.

7) It does hold that $c \cdot (\alpha \oplus \beta) = c \cdot \alpha \oplus c \cdot \beta$. Firstly, $c \cdot (\alpha \oplus \beta) = c \cdot (\alpha - \beta) = -c(\alpha - \beta) = -c\alpha + c\beta$. And secondly $c \cdot \alpha \oplus c \cdot \beta = (-c\alpha) \oplus (-c\beta) = -c\alpha - (-c\beta) = -c\alpha + c\beta$. Thus they are equal.

8) It does not hold that $(c_1 + c_2) \cdot \alpha = (c_1 \cdot \alpha) \oplus (c_2 \cdot \alpha)$. Firstly, $(c_1 + c_2) \cdot \alpha = -(c_1 + c_2) \alpha = -c_1 \alpha - c_2 \alpha$. Secondly, $c_1 \cdot \alpha \oplus c_2 \cdot \alpha = (-c_1 \cdot \alpha) \oplus (-c_2 \cdot \alpha) = -c_1 \alpha + c_2 \alpha$. Since $-c_1 \alpha - c_2 \alpha \neq -c_1 \alpha + c_2 \alpha$ for all c_1, c_2 they are not equal.

Exercise 6: Let V be the set of all complex-valued functions f on the real line such that (for all t in \mathbb{R})

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that V , with the operations

$$(f + g)(t) = f(t) + g(t)$$

$$(cf)(t) = cf(t)$$

is a vector space over the field of real numbers. Give an example of a function in V which is not real-valued.

Solution: We will use the basic fact that $\overline{a + b} = \overline{a} + \overline{b}$ and $\overline{ab} = \overline{a} \cdot \overline{b}$.

Before we show V satisfies the eight properties we must first show vector addition and scalar multiplication as defined are actually well-defined in the sense that they are indeed operations on V . In other words if f and g are two functions in V then we must show that $f + g$ is in V . In other words if $f(-t) = \overline{f(t)}$ and $g(-t) = \overline{g(t)}$ then we must show that $(f + g)(-t) = \overline{(f + g)(t)}$. This is true because $(f + g)(-t) = f(-t) + g(-t) = \overline{f(t)} + \overline{g(t)} = \overline{f(t) + g(t)} = \overline{(f + g)(t)}$.

Similarly, if $c \in \mathbb{R}$, $(cf)(-t) = cf(-t) = \overline{cf(t)} = \overline{cf(t)}$ since $\overline{c} = c$ when $c \in \mathbb{R}$.

Thus the operations are well defined. We now show the eight properties hold:

1) Addition on functions in V is defined by adding in \mathbb{C} to the values of the functions in \mathbb{C} . Thus since \mathbb{C} is commutative, addition in V inherits this commutativity.

2) Similar to 1, since \mathbb{C} is associative, addition in V inherits this associativity.

3) The zero function $g(t) = 0$ is in V since $-0 = \overline{0}$. And g satisfies $f + g = f$ for all $f \in V$. Thus V has a right additive identity.

4) Let g be the function $g(t) = -f(t)$. Then $g(-t) = -f(-t) = -\overline{-f(t)} = \overline{-f(t)} = \overline{g(t)}$. Thus $g \in V$. And $(f + g)(t) = f(t) + g(t) = f(t) - f(t) = 0$. Thus g is a right additive inverse for f .

5) Clearly $1 \cdot f = f$ since 1 is the multiplicative identity in \mathbb{R} .

6) As before, associativity in \mathbb{C} implies $(c_1 c_2) f = c_1 (c_2 f)$.

7) Similarly, the distributive property in \mathbb{C} implies $c(f + g) = cf + cg$.

8) Similarly, the distributive property in \mathbb{C} implies $(c_1 + c_2)f = c_1 f + c_2 f$.

An example of a function in V which is not real valued is $f(x) = ix$. Since $f(1) = i$ is not real-valued. And $f(-x) = -ix = \overline{ix}$ since $x \in \mathbb{R}$, so $f \in V$.

Exercise 7: Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$

$$c(x, y) = (cx, 0).$$

Is V , with these operations, a vector space?

Solution: This is not a vector space because there would have to be an additive identity element (a, b) which has the property that $(a, b) + (x, y) = (x, y)$ for all $(x, y) \in V$. But this is impossible, because $(a, b) + (0, 1) = (a, 0) \neq (0, 1)$ no matter what (a, b) is. Thus V does not satisfy the third requirement of having an additive identity element.

Section 2.2: Subspaces

Page 39, typo in Exercise 3. It says \mathbb{R}^5 , should be \mathbb{R}^4 .

Exercise 1: Which of the following sets of vectors $\alpha = (a_1, \dots, a_n)$ in \mathbb{R}^n are subspaces of \mathbb{R}^n ($n \geq 3$)?

- (a) all α such that $a_1 \geq 0$;
- (b) all α such that $a_1 + 3a_2 = a_3$;
- (c) all α such that $a_2 = a_1^2$;
- (d) all α such that $a_1 a_2 = 0$;
- (e) all α such that a_2 is rational.

Solution:

(a) This is not a subspace because for $(1, \dots, 1)$ the additive inverse is $(-1, \dots, -1)$ which does not satisfy the condition.

(b) Suppose $(a_1, a_2, a_3, \dots, a_n)$ and $(b_1, b_2, b_3, \dots, b_n)$ satisfy the condition and let $c \in \mathbb{R}$. By Theorem 1 (page 35) we must show that $c(a_1, a_2, a_3, \dots, a_n) + (b_1, b_2, b_3, \dots, b_n) = (ca_1 + b_1, \dots, ca_n + b_n)$ satisfies the condition. Now $(ca_1 + b_1) + 3(ca_2 + b_2) = c(a_1 + 3a_2) + (b_1 + 3b_2) = c(a_3) + (b_3) = ca_3 + b_3$. Thus it does satisfy the condition so V is a vector space.

(c) This is not a vector space because $(1, 1)$ satisfies the condition since $1^2 = 1$, but $(1, 1, \dots) + (1, 1, \dots) = (2, 2, \dots)$ and $(2, 2, \dots)$ does not satisfy the condition because $2^2 \neq 2$.

(d) This is not a subspace. $(1, 0, \dots)$ and $(0, 1, \dots)$ both satisfy the condition, but their sum is $(1, 1, \dots)$ which does not satisfy the condition.

(e) This is not a subspace. $(1, 1, \dots, 1)$ satisfies the condition, but $\pi(1, 1, \dots, 1) = (\pi, \pi, \dots, \pi)$ does not satisfy the condition.

Exercise 2: Let V be the (real) vector space of all functions f from \mathbb{R} into \mathbb{R} . Which of the following sets of functions are subspaces of V ?

(a) all f such that $f(x^2) = f(x)^2$;

(b) all f such that $f(0) = f(1)$;

(c) all f such that $f(3) = 1 + f(-5)$;

(d) all f such that $f(-1) = 0$;

(e) all f which are continuous.

Solution:

(a) Not a subspace. Let $f(x) = x$ and $g(x) = x^2$. Then both satisfy the condition: $f(x^2) = x^2 = (f(x))^2$ and $g(x^2) = (x^2)^2 = (g(x))^2$. But $(f + g)(x) = x + x^2$ and $(f + g)(x^2) = x^2 + x^4$ while $[(f + g)(x)]^2 = (x + x^2)^2 = x^4 + 2x^3 + x^2$. These are not equal polynomials so the condition does not hold for $f + g$.

(b) Yes a subspace. Suppose f and g satisfy the property. Let $c \in \mathbb{R}$. Then $(cf + g)(0) = cf(0) + g(0) = cf(1) + g(1) = (cf + g)(1)$. Thus $(cf + g)(0) = (cf + g)(1)$. By Theorem 1 (page 35) the set of all such functions constitute a subspace.

(c) Not a subspace. Let $f(x)$ be the function defined by $f(3) = 1$ and $f(x) = 0$ for all $x \neq 3$. Let $g(x)$ be the function defined by $g(-5) = 0$ and $g(x) = 1$ for all $x \neq -5$. Then both f and g satisfy the condition. But $(f + g)(3) = f(3) + g(3) = 1 + 1 = 2$, while $1 + (f + g)(-5) = 1 + f(-5) + g(-5) = 1 + 0 + 0 = 1$. Since $1 \neq 2$, $f + g$ does not satisfy the condition.

(d) Yes a subspace. Suppose f and g satisfy the property. Let $c \in \mathbb{R}$. Then $(cf + g)(-1) = cf(-1) + g(-1) = c \cdot 0 + 0 = 0$. Thus $(cf + g)(-1) = 0$. By Theorem 1 (page 35) the set of all such functions constitute a subspace.

(e) Yes a subspace. Let f and g be continuous functions from \mathbb{R} to \mathbb{R} and let $c \in \mathbb{R}$. Then we know from basic results of real analysis that the sum and product of continuous functions are continuous. Since the function $c \mapsto c$ is continuous as well as f and g , it follows that $cf + g$ is continuous. By Theorem 1 (page 35) the set of all continuous functions constitute a subspace.

Exercise 3: Is the vector $(3, -1, 0, -1)$ in the subspace of \mathbb{R}^5 (sic) spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, -5)$?

Solution: I assume they meant \mathbb{R}^4 . No, $(3, -1, 0, -1)$ is not in the subspace. If we row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{array} \right]$$

we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & -2 \end{array} \right].$$

The two bottom rows are zero rows to the left of the divider, but the values to the right of the divider in those two rows are non-zero. Thus the system does not have a solution (see comments bottom of page 24 and top of page 25).

Exercise 4: Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in \mathbb{R}^5 which satisfy

$$\begin{aligned} 2x_1 - x_2 + \frac{4}{3}x_3 - x_4 &= 0 \\ x_1 + \frac{2}{3}x_3 - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0. \end{aligned}$$

Find a finite set of vectors which spans W .

Solution: The matrix of the system is

$$\begin{bmatrix} 2 & -1 & 4/3 & -1 & 0 \\ 1 & 0 & 2/3 & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{bmatrix}.$$

Row reducing to reduced echelon form gives

$$\begin{bmatrix} 1 & 0 & 2/3 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the system is equivalent to

$$\begin{aligned} x_1 + 2/3x_3 - x_5 &= 0 \\ x_2 + x_4 - 2x_5 &= 0. \end{aligned}$$

Thus the system is parametrized by (x_3, x_4, x_5) . Setting each equal to one and the other two to zero, in turn, gives the three vectors $(-2/3, 0, 1, 0, 0)$, $(0, -1, 0, 1, 0)$ and $(1, 2, 0, 0, 1)$. These three vectors therefore span W .

Exercise 5: Let F be a field and let n be a positive integer ($n \geq 2$). Let V be the vectors space of all $n \times n$ matrices over F . Which of the following sets of matrices A in V are subspaces of V ?

- (a) all invertible A ;
- (b) all non-invertible A ;
- (c) all A such that $AB = BA$, where B is some fixed matrix in V ;
- (d) all A such that $A^2 = A$.

Solution:

(a) This is not a subspace. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then both A and B are invertible, but $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is not invertible. Thus the subset is not closed with respect to matrix addition. Therefore it cannot be a subspace.

(b) This is not a subspace. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then neither A nor B is invertible, but $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is invertible. Thus the subset is not closed with respect to matrix addition. Therefore it cannot be a subspace.

(c) This is a subspace. Suppose A_1 and A_2 satisfy $A_1B = BA_1$ and $A_2B = BA_2$. Let $c \in F$ be any constant. Then $(cA_1 + A_2)B = cA_1B + A_2B = cBA_1 + BA_2 = B(cA_1) + BA_2 = B(cA_1 + A_2)$. Thus $cA_1 + A_2$ satisfy the criteria. By Theorem 1 (page 35) the subset is a subspace.

(d) This is a subspace if F equals $\mathbb{Z}/2\mathbb{Z}$, otherwise it is not a subspace.

Suppose first that $\text{char}(F) \neq 2$. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A^2 = A$. But $A + A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Thus $A + A$ does not satisfy the criteria so the subset cannot be a subspace.

Suppose now that $F = \mathbb{Z}/2\mathbb{Z}$. Suppose A and B both satisfy $A^2 = A$ and $B^2 = B$. Let $c \in F$ be any scalar. Then $(cA + B)^2 = c^2A^2 + 2cAB + B^2$. Now $2 = 0$ so this reduces to $c^2A^2 + B^2$. If $c = 0$ then this reduces to B^2 which equals B , if $c = 1$ then this reduces to $A^2 + B^2$ which equals $A + B$. In both cases $(cA + B)^2 = cA + B$. Thus in this case by Theorem 1 (page 35) the subset is a subspace.

Finally suppose $\text{char}(F) = 2$ but F is not $\mathbb{Z}/2\mathbb{Z}$. Then $|F| > 2$. The polynomial $x^2 - x = 0$ has at most two solutions in F , so $\exists c \in F$ such that $c^2 \neq c$. Consider the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $I^2 = I$. If such matrices form a subspace then it must be that cI is also in the subspace. Thus it must be that $(cI)^2 = cI$. Which is equivalent to $c^2 = c$, which contradicts the way c was chosen.

Exercise 6:

- Prove that the only subspaces of \mathbb{R}^1 are \mathbb{R}^1 and the zero subspace.
- Prove that a subspace of \mathbb{R}^2 is \mathbb{R}^2 , or the zero subspace, or consists of all scalar multiples of some fixed vector in \mathbb{R}^2 . (The last type of subspace is, intuitively, a straight line through the origin.)
- Can you describe the subspaces of \mathbb{R}^3 ?

Solution:

(a) Let V be a subspace of \mathbb{R}^1 . Suppose $\exists v \in V$ with $v \neq 0$. Then v is a vector but it is also simply an element of \mathbb{R} . Let $\alpha \in \mathbb{R}$. Then $\alpha = \frac{\alpha}{v} \cdot v$ where $\frac{\alpha}{v}$ is a scalar in the base field \mathbb{R} . Since $cv \in V \forall c \in \mathbb{R}$, it follows that $\alpha \in V$. Thus we have shown that if $V \neq \{0\}$ then $V = \mathbb{R}^1$.

(b) We know the subsets $\{(0, 0)\}$ (example 6a, page 35) and \mathbb{R}^2 (example 1, page 29) are subspaces of \mathbb{R}^2 . Also for any vector v in any vector space over any field F , the set $\{cv \mid c \in F\}$ is a subspace (Theorem 3, page 37). Thus we must show that if V is a subspace of \mathbb{R}^2 and there exists $v_1, v_2 \in V$ such that v_1 and v_2 do not lie on the same line, then $V = \mathbb{R}^2$. Equivalently we must show that any vector $w \in \mathbb{R}^2$ can be written as a linear combination of v_1 and v_2 whenever v_1 and v_2 are not co-linear. Equivalently, by Theorem 13 (iii) (page 23), it suffices to show that if $v_1 = (a, b)$ and $v_2 = (c, d)$ are not colinear, then the matrix $A = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}$ is invertible. Suppose $a \neq 0$ and let $x = c/a$. Then $xa = c$, and since v_1 and v_2 are not colinear, it follows that $xb \neq d$. Thus equivalently $ad - bc \neq 0$. It follows now from Exercise 1.6.8 page 27 that if v_1 and v_2 not colinear then the matrix A^T is invertible. Finally A^T is invertible implies A is invertible, since clearly $(A^T)^{-1} = (A^{-1})^T$. Similarly if $a = 0$ then it must be that $b \neq 0$ so we can make the same argument. So in all cases A is invertible.

(c) The subspaces are the zero subspace $\{0, 0, 0\}$, lines $\{cv \mid c \in \mathbb{R}\}$ for fixed $v \in \mathbb{R}^3$, planes $\{c_1v_1 + c_2v_2 \mid c_1, c_2 \in \mathbb{R}\}$ for fixed $v_1, v_2 \in \mathbb{R}^3$ and the whole space \mathbb{R}^3 . By Theorem 3 we know these all are subspaces, we just must show they are the only subspaces. It suffices to show that if v_1, v_2 and v_3 are not co-planar then the space generated by v_1, v_2, v_3 is all of \mathbb{R}^3 . Equivalently we must show if v_3 is not in the plane generated by v_1, v_2 then any vector $w \in \mathbb{R}^3$ can be written as a linear combination of v_1, v_2, v_3 . Equivalently, by Theorem 13 (iii) (page 23), it suffices to show the matrix $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ is invertible. A is invertible $\Leftrightarrow A^T$ is invertible \Leftrightarrow since clearly $(A^T)^{-1} = (A^{-1})^T$. Now v_3 is in the plane generated by $v_1, v_2 \Leftrightarrow v_3$ can be written as a linear combination of v_1 and $v_2 \Leftrightarrow A^T$ is row equivalent to a matrix with a row equal to all zeros (this follows from Theorem 12, page 23) $\Leftrightarrow A^T$ is not invertible. Thus v_3 is not in the plane generated by $v_1, v_2 \Leftrightarrow A$ is invertible.

Exercise 7: Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

Solution: Assume the space generated by W_1 and W_2 is equal to their set-theoretic union $W_1 \cup W_2$. Suppose $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. We wish to derive a contradiction. So suppose $\exists w_1 \in W_1 \setminus W_2$ and $\exists w_2 \in W_2 \setminus W_1$. Consider $w_1 + w_2$. By assumption this is in $W_1 \cup W_2$, so $\exists w'_1 \in W_1$ such that $w_1 + w_2 = w'_1$ or $\exists w'_2 \in W_2$ such that $w_1 + w_2 = w'_2$. If the former, then $w_2 = w'_1 - w_1 \in W_1$ which contradicts the assumption that $w_2 \notin W_1$. Likewise the latter implies the contradiction $w_1 \in W_2$. Thus we are done.

Exercise 8: Let V be the vector space of all functions from \mathbb{R} into \mathbb{R} ; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.

- Prove that V_e and V_o are subspaces of V .
- Prove that $V_e + V_o = V$.
- Prove that $V_e \cap V_o = \{0\}$.

Solution:

(a) Let $f, g \in V_e$ and $c \in \mathbb{R}$. Let $h = cf + g$. Then $h(-x) = cf(-x) + g(-x) = cf(x) + g(x) = h(x)$. So $h \in V_e$. By Theorem 1 (page 35) V_e is a subspace. Now let $f, g \in V_o$ and $c \in \mathbb{R}$. Let $h = cf + g$. Then $h(-x) = cf(-x) + g(-x) = -cf(x) - g(x) = -h(x)$. So $h \in V_o$. By Theorem 1 (page 35) V_o is a subspace.

(b) Let $f \in V$. Let $f_e(x) = \frac{f(x) + f(-x)}{2}$ and $f_o = \frac{f(x) - f(-x)}{2}$. Then f_e is even and f_o is odd and $f = f_e + f_o$. Thus we have written f as $g + h$ where $g \in V_e$ and $h \in V_o$.

(c) Let $f \in V_e \cap V_o$. Then $f(-x) = -f(x)$ and $f(-x) = f(x)$. Thus $f(x) = -f(x)$ which implies $2f(x) = 0$ which implies $f = 0$.

Exercise 9: Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are *unique* vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

Solution: Let $\alpha \in W$. Suppose $\alpha = \alpha_1 + \alpha_2$ for $\alpha_i \in W_i$ and $\alpha = \beta_1 + \beta_2$ for $\beta_i \in W_i$. Then $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ which implies $\alpha_1 - \beta_1 = \beta_2 - \alpha_2$. Thus $\alpha_1 - \beta_1 \in W_1$ and $\alpha_1 - \beta_1 \in W_2$. Since $W_1 \cap W_2 = \{0\}$ it follows that $\alpha_1 - \beta_1 = 0$ and thus $\alpha_1 = \beta_1$. Similarly, $\beta_2 - \alpha_2 \in W_1 \cap W_2$ so also $\alpha_2 = \beta_2$.

Section 2.3: Bases and Dimension

Exercise 1: Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

Solution: Suppose v_1 and v_2 are linearly dependent. Then $\exists c_1, c_2$ such that $c_1v_1 + c_2v_2 = 0$ and at least one of c_1 or c_2 is non-zero. Assume WLOG that $c_1 \neq 0$. Then $v_1 = -\frac{c_2}{c_1}v_2$.

Exercise 2: Are the vectors

$$\alpha_1 = (1, 1, 2, 4), \quad \alpha_2 = (2, -1, -5, 2)$$

$$\alpha_3 = (1, -1, -4, 0), \quad \alpha_4 = (2, 1, 1, 6)$$

linearly independent in \mathbb{R}^4 ?

Solution: By Corollary 3, page 46, it suffices to determine if the matrix whose rows are the α_i 's is invertible. By Theorem 12 (ii) we can do this by row reducing the matrix

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the four vectors are not linearly independent.

Exercise 3: Find a basis for the subspace of \mathbb{R}^4 spanned by the four vectors of Exercise 2.

Solution: In Section 2.5, Theorem 9, page 56, it will be proven that row equivalent matrices have the same row space. The proof of this is almost immediate so there seems no easier way to prove it than to use that fact. If you multiply a matrix A on the left by another matrix P , the rows of the new matrix PA are linear combinations of the rows of the original matrix. Thus the rows of PA generate a subspace of the space generated by the rows of A . If P is invertible, then the two spaces must be contained in each other since we can go backwards with P^{-1} . Thus the rows of row-equivalent matrices generate the same space. Thus using the row reduced form of the matrix in Exercise 2, it must be that the space is two dimensional and generated by $(1, 1, 2, 4)$ and $(0, 1, 3, 2)$.

Exercise 4: Show that the vectors

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 2, 1), \quad \alpha_3 = (0, -3, 2)$$

form a basis for \mathbb{R}^3 . Express each of the standard basis vectors as linear combinations of α_1 , α_2 , and α_3 .

Solution: By Corollary 3, page 46, to show the vectors are linearly independent it suffices to show the matrix whose rows are the α_i 's is invertible. By Theorem 12 (ii) we can do this by row reducing the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & -3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now to write the standard basis vectors in terms of these vectors, by the discussion at the bottom of page 25 through page 26, we can row-reduce the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{array} \right].$$

This gives

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 5 & -3/2 & 3/2 & 1 \end{array} \right] \end{aligned}$$

$$\begin{aligned} &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -3/10 & 3/10 & 1/5 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/10 & 3/10 & 1/5 \\ 0 & 1 & 0 & -1/5 & 1/5 & -1/5 \\ 0 & 0 & 1 & -3/10 & 3/10 & 1/5 \end{array} \right]. \end{aligned}$$

Thus if

$$P = \begin{bmatrix} 7/10 & 3/10 & 1/5 \\ -1/5 & 1/5 & -1/5 \\ -3/10 & 3/10 & 1/5 \end{bmatrix}$$

then $PA = I$, so we have

$$\begin{aligned} \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 &= (1, 0, 0) \\ -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 &= (0, 1, 0) \\ -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 &= (0, 0, 1). \end{aligned}$$

Exercise 5: Find three vectors in \mathbb{R}^3 which are linearly dependent, and are such that any two of them are linearly independent.

Solution: Let $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$ and $v_3 = (1, 1, 0)$. Then $v_1 + v_2 - v_3 = (0, 0, 0)$ so they are linearly dependent. We know v_1 and v_2 are linearly independent as they are two of the standard basis vectors (see Example 13, page 41). Suppose $av_1 + bv_3 = 0$. Then $(a + b, b, 0) = (0, 0, 0)$. The second coordinate implies $b = 0$ and then the first coordinate in turn implies $a = 0$. Thus v_1 and v_3 are linearly independent. Analogously v_2 and v_3 are linearly independent.

Exercise 6: Let V be the vector space of all 2×2 matrices over the field F . Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

Solution: Let

$$\begin{aligned} v_{11} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & v_{12} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ v_{21} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, & v_{22} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Suppose $av_{11} + bv_{12} + cv_{21} + dv_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

from which it follows immediately that $a = b = c = d = 0$. Thus $v_{11}, v_{12}, v_{21}, v_{22}$ are linearly independent.

Now let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any 2×2 matrix. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = av_{11} + bv_{12} + cv_{21} + dv_{22}$. Thus $v_{11}, v_{12}, v_{21}, v_{22}$ span the space of 2×2 matrices.

Thus $v_{11}, v_{12}, v_{21}, v_{22}$ are both linearly independent and they span the space of all 2×2 matrices. Thus $v_{11}, v_{12}, v_{21}, v_{22}$ constitute a basis for the space of all 2×2 matrices.

Exercise 7: Let V be the vector space of Exercise 6. Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

- (a) Prove that W_1 and W_2 are subspaces of V .
 (b) Find the dimension of W_1 , W_2 , $W_1 + W_2$, and $W_1 \cap W_2$.

Solution:

(a) Let $A = \begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and $B = \begin{bmatrix} x' & -x' \\ y' & z' \end{bmatrix}$ be two elements of W_1 and let $c \in F$. Then

$$cA + B = \begin{bmatrix} cx + x' & -cx - x' \\ cy + y' & cz + z' \end{bmatrix} = \begin{bmatrix} a & -a \\ u & v \end{bmatrix}$$

where $a = cx + x'$, $u = cy + y'$ and $v = cz + z'$. Thus $cA + B$ is in the form of an element of W_1 . Thus $cA + B \in W_1$. By Theorem 1 (page 35) W_1 is a subspace.

Now let $A = \begin{bmatrix} a & b \\ -a & d \end{bmatrix}$ and $B = \begin{bmatrix} a' & b' \\ -a' & d' \end{bmatrix}$ be two elements of W_2 and let $c \in F$. Then

$$cA + B = \begin{bmatrix} ca + a' & cb + b' \\ -ca - a' & cd + d' \end{bmatrix} = \begin{bmatrix} x & y \\ -x & z \end{bmatrix}$$

where $x = ca + a'$, $y = cb + b'$ and $z = cd + d'$. Thus $cA + B$ is in the form of an element of W_2 . Thus $cA + B \in W_2$. By Theorem 1 (page 35) W_2 is a subspace.

(b) Let

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $A_1, A_2, A_3 \in W_1$ and

$$c_1A_1 + c_2A_2 + c_3A_3 = \begin{bmatrix} c_1 & -c_1 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

implies $c_1 = c_2 = c_3 = 0$. So A_1, A_2, A_3 are linearly independent. Now let $A = \begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ be any element of W_1 . Then $A = xA_1 + yA_2 + zA_3$. Thus A_1, A_2, A_3 span W_1 . Thus $\{A_1, A_2, A_3\}$ form a basis for W_1 . Thus W_1 has dimension three.

Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $A_1, A_2, A_3 \in W_2$ and

$$c_1A_1 + c_2A_2 + c_3A_3 = \begin{bmatrix} c_1 & c_2 \\ -c_1 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

implies $c_1 = c_2 = c_3 = 0$. So A_1, A_2, A_3 are linearly independent. Now let $A = \begin{bmatrix} x & y \\ -x & z \end{bmatrix}$ be any element of W_2 . Then $A = xA_1 + yA_2 + zA_3$. Thus A_1, A_2, A_3 span W_2 . Thus $\{A_1, A_2, A_3\}$ form a basis for W_2 . Thus W_2 has dimension three.

Let V be the space of 2×2 matrices. We showed in Exercise 6 that the $\dim(V) = 4$. Now $W_1 \subseteq W_1 + W_2 \subseteq V$. Thus by Corollary 1, page 46, $3 \leq \dim(W_1 + W_2) \leq 4$. Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Then $A \in W_2$ and $A \notin W_1$. Thus $W_1 + W_2$ is strictly bigger

than W_1 . Thus $4 \geq \dim(W_1 + W_2) > \dim(W_1) = 3$. Thus $\dim(W_1 + W_2) = 4$.

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $W_1 \cap W_2$. Then $A \in W_1 \Rightarrow a = -b$ and $A \in W_2 \Rightarrow a = -c$. So $A = \begin{bmatrix} a & -a \\ -a & b \end{bmatrix}$. Let $A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Suppose $aA_1 + bA_2 = 0$. Then

$$\begin{bmatrix} a & -a \\ -a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which implies $a = b = 0$. Thus A_1 and A_2 are linearly independent. Let $A = \begin{bmatrix} a & -a \\ -a & b \end{bmatrix} \in W_1 \cap W_2$. Then $A = aA_1 + bA_2$. So A_1, A_2 span $W_1 \cap W_2$. Thus $\{A_1, A_2\}$ is a basis for $W_1 \cap W_2$. Thus $\dim(W_1 \cap W_2) = 2$.

Exercise 8: Again let V be the space of 2×2 matrices over F . Find a basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A_j^2 = A_j$ for each j .

Solution: Let V be the space of all 2×2 matrices. Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Then $A_i^2 = A_i \forall i$. Now

$$aA_1 + bA_2 + cA_3 + dA_4 = \begin{bmatrix} a+c & c \\ d & b+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

implies $c = d = 0$ which in turn implies $a = b = 0$. Thus A_1, A_2, A_3, A_4 are linearly independent. Thus they span a subspace of A of dimension four. But by Exercise 6, A also has dimension four. Thus by Corollary 1, page 46, the subspace spanned by A_1, A_2, A_3, A_4 is the entire space. Thus $\{A_1, A_2, A_3, A_4\}$ is a basis.

Exercise 9: Let V be a vector space over a subfield F of the complex numbers. Suppose α, β , and γ are linearly independent vectors in V . Prove that $(\alpha + \beta)$, $(\beta + \gamma)$, and $(\gamma + \alpha)$ are linearly independent.

Solution: Suppose $a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0$. Rearranging gives $(a + c)\alpha + (a + b)\beta + (b + c)\gamma = 0$. Since α, β , and γ are linearly independent it follows that $a + c = a + b = b + c = 0$. This gives a system of equations in a, b, c with matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

This row-reduces as follows:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since this row-reduces to the identity matrix, by Theorem 7, page 13, the only solution is $a = b = c = 0$. Thus $(\alpha + \beta)$, $(\beta + \gamma)$, and $(\gamma + \alpha)$ are linearly independent.

Exercise 10: Let V be a vector space over the field F . Suppose there are a finite number of vectors $\alpha_1, \dots, \alpha_r$ in V which span V . Prove that V is finite-dimensional.

Solution: If any α_i 's are equal to zero then we can remove them from the set and the remaining α_i 's still span V . Thus we can assume WLOG that $\alpha_i \neq 0 \forall i$. If $\alpha_1, \dots, \alpha_r$ are linearly independent, then $\{\alpha_1, \dots, \alpha_r\}$ is a basis and $\dim(V) = r < \infty$. On the other hand if $\alpha_1, \dots, \alpha_r$ are linearly dependent, then $\exists c_1, \dots, c_r \in F$, not all zero, such that $c_1\alpha_1 + \dots + c_r\alpha_r = 0$. Suppose WLOG that $c_r \neq 0$. Then $\alpha_r = -\frac{c_1}{c_r}\alpha_1 - \dots - \frac{c_{r-1}}{c_r}\alpha_{r-1}$. Thus α_r is in the subspace spanned by $\alpha_1, \dots, \alpha_{r-1}$. Thus $\alpha_1, \dots, \alpha_{r-1}$ spans V . If $\alpha_1, \dots, \alpha_{r-1}$ are linearly independent then $\{\alpha_1, \dots, \alpha_{r-1}\}$ is a basis and $\dim(V) = r - 1 < \infty$. If $\alpha_1, \dots, \alpha_{r-1}$ are linearly dependent then arguing as before (with possibly re-indexing) we can produce $\alpha_1, \dots, \alpha_{r-2}$ which span V . Continuing in this way we must eventually arrive at a linearly independent set, or arrive at a set that consists of a single element, that still spans V . If we arrive at a single element v_1 then $\{v_1\}$ is linearly independent since $cv_1 = 0 \Rightarrow c = 0$ (see comments after (2-9) page 31). Thus we must eventually arrive at a finite set that spans and is linearly independent. Thus we must eventually arrive at a finite basis, which implies $\dim(V) < \infty$.

Exercise 11: Let V be the set of all 2×2 matrices A with *complex* entries which satisfy $A_{11} + A_{22} = 0$.

- Show that V is a vector space over the field of *real* numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.
- Find a basis for this vector space.
- Let W be the set of all matrices A in V such that $A_{21} = -\overline{A_{12}}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W .

Solution: (a) It is clear from inspection of the definition of a vector space (pages 28-29) that a vector space over a field F is a vector space over every subfield of F , because all properties (e.g. commutativity and associativity) are inherited from the operations in F . Let M be the space of all 2×2 matrices over \mathbb{C} . We will show M is a subspace V as a vector space over \mathbb{C} . It will follow from the comment above that M is a vector space over \mathbb{R} . Now V is a subset of M , so using Theorem 1 (page 35) we must show whenever $A, B \in V$ and $c \in \mathbb{C}$ then $cA + B \in V$. Let $A, B \in V$. Write $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and $B = \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}$. Then

$$x + w = x' + w' = 0. \quad (17)$$

$$cA + B = \begin{bmatrix} cx + x' & cy + y' \\ cz + z' & cw + w' \end{bmatrix}$$

To show $cA + B \in V$ we must show $(cx + x') + (cw + w') = 0$. Rearranging the left hand side gives $c(x + w) + (x' + w')$ which equals zero by (17).

(b) We can write the general element of V as

$$A = \begin{bmatrix} a + bi & e + fi \\ g + hi & -a - bi \end{bmatrix}.$$

Let

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

$$v_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix},$$

$$v_5 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}.$$

Then $A = av_1 + bv_2 + ev_3 + fv_4 + gv_5 + hv_6$ so $v_1, v_2, v_3, v_4, v_5, v_6$ span V . Suppose $av_1 + bv_2 + ev_3 + fv_4 + gv_5 + hv_6 = 0$. Then

$$av_1 + bv_2 + ev_3 + fv_4 + gv_5 + hv_6 = \begin{bmatrix} a + bi & e + fi \\ g + hi & -a - bi \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

implies $a = b = c = d = e = f = g = h = 0$ because a complex number $u + vi = 0 \Leftrightarrow u = v = 0$. Thus $v_1, v_2, v_3, v_4, v_5, v_6$ are linearly independent. Thus $\{v_1, \dots, v_6\}$ is a basis for V as a vector space over \mathbb{R} , and $\dim(V) = 6$.

(c) Let $A, B \in W$ and $c \in \mathbb{R}$. By Theorem 1 (page 35) we must show $cA + B \in W$. Write $A = \begin{bmatrix} x & y \\ z & -\bar{x} \end{bmatrix}$ and $B = \begin{bmatrix} x' & y' \\ z' & -\bar{x}' \end{bmatrix}$, where $x, y, z, x', y', z' \in \mathbb{C}$. Then

$$cA + B = \begin{bmatrix} cx + x' & cy + y' \\ cz + z' & -c\bar{x} - \bar{x}' \end{bmatrix}.$$

Since $-c\bar{x} - \bar{x}' = -\overline{(cx + x')}$, it follows that $cA + B \in W$.

Exercise 12: Prove that the space of $m \times n$ matrices over the field F has dimension mn , by exhibiting a basis for this space.

Solution: Let M be the space of all $m \times n$ matrices. Let M_{ij} be the matrix of all zeros except for the i, j -th place which is a one. We claim $\{M_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ constitute a basis for M . Let $A = (a_{ij})$ be an arbitrary matrix in M . Then $A = \sum_{ij} a_{ij}M_{ij}$. Thus $\{M_{ij}\}$ span M . Suppose $\sum_{ij} a_{ij}M_{ij} = 0$. The left hand side equals the matrix (a_{ij}) and this equals the zero matrix if and only if every $a_{ij} = 0$. Thus $\{M_{ij}\}$ are linearly independent as well. Thus they constitute a basis and M has dimension mn .

Exercise 13: Discuss Exercise 9, when V is a vector space over the field with two elements described in Exercise 5, Section 1.1.

Solution: If F has characteristic two then $(\alpha + \beta) + (\beta + \gamma) + (\gamma + \alpha) = 2\alpha + 2\beta + 2\gamma = 0 + 0 + 0 = 0$ since in a field of characteristic two, $2 = 0$. Thus in this case $(\alpha + \beta)$, $(\beta + \gamma)$ and $(\gamma + \alpha)$ are linearly dependent.

Exercise 14: Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.

Solution: We know that \mathbb{Q} is countable and \mathbb{R} is uncountable. Since the set of n -tuples of things from a countable set is countable, \mathbb{Q}^n is countable for all n . Now, suppose $\{r_1, \dots, r_n\}$ is a basis for \mathbb{R} over \mathbb{Q} . Then every element of \mathbb{R} can be written as $a_1r_1 + \dots + a_nr_n$. Thus we can map n -tuples of rational numbers onto \mathbb{R} by $(a_1, \dots, a_n) \mapsto a_1r_1 + \dots + a_nr_n$. Thus the cardinality of \mathbb{R} must be less or equal to \mathbb{Q}^n . But the former is uncountable and the latter is countable, a contradiction. Thus there can be no such finite basis.

Section 2.4: Coordinates

Exercise 1: Show that the vectors

$$\alpha_1 = (1, 1, 0, 0), \quad \alpha_2 = (0, 0, 1, 1)$$

$$\alpha_3 = (1, 0, 0, 4), \quad \alpha_4 = (0, 0, 0, 2)$$

form a basis for \mathbb{R}^4 . Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Solution: Using Theorem 7, page 52, if we calculate the inverse of

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{bmatrix}.$$

then the columns of P^{-1} will give the coefficients to write the standard basis vectors in terms of the α_i 's. We do this by

row-reducing the augmented matrix

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{array} \right].$$

The left side must reduce to the identity while the right side transforms to the inverse of P . Row reduction gives

$$\begin{aligned} & \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & -1/2 & 1/2 \end{array} \right]. \end{aligned}$$

Thus $\{\alpha_1, \dots, \alpha_4\}$ is a basis. Call this basis β . Thus $(1, 0, 0, 0) = \alpha_3 - 2\alpha_4$, $(0, 1, 0, 0) = \alpha_1 - \alpha_3 + 2\alpha_4$, $(0, 0, 1, 0) = \alpha_2 - \frac{1}{2}\alpha_4$ and $(0, 0, 0, 1) = \frac{1}{2}\alpha_4$.

Thus $[(1, 0, 0, 0)]_\beta = (0, 0, 1, -2)$, $[(0, 1, 0, 0)]_\beta = (1, 0, -1, 2)$, $[(0, 0, 1, 0)]_\beta = (0, 1, 0, -1/2)$ and $[(0, 0, 0, 1)]_\beta = (0, 0, 0, 1/2)$.

Exercise 2: Find the coordinate matrix of the vector $(1, 0, 1)$ in the basis of \mathbb{C}^3 consisting of the vectors $(2i, 1, 0)$, $(2, -1, 0)$, $(0, 1 + i, 1 - i)$, in that order.

Solution: Using Theorem 7, page 52, the answer is $P^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ where

$$P = \begin{bmatrix} 2i & 2 & 0 \\ 1 & -1 & 1+i \\ 0 & 0 & 1-i \end{bmatrix}.$$

We find P^{-1} by row-reducing the augmented matrix

$$\left[\begin{array}{ccc|ccc} 2i & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1+i & 0 & 1 & 0 \\ 0 & 0 & 1-i & 0 & 0 & 1 \end{array} \right].$$

The right side will transform into the P^{-1} . Row reducing:

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 2i & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1+i & 0 & 1 & 0 \\ 0 & 0 & 1-i & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1+i & 0 & 1 & 0 \\ 2i & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1-i & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1+i & 0 & 1 & 0 \\ 0 & 2+2i & 2-2i & 1 & -2i & 0 \\ 0 & 0 & 1-i & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1+i & 0 & 1 & 0 \\ 0 & 1 & -i & \frac{1-i}{4} & \frac{-1-i}{2} & 0 \\ 0 & 0 & 1-i & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1-i}{4} & \frac{1-i}{2} & 0 \\ 0 & 1 & -i & \frac{1-i}{4} & \frac{-1-i}{2} & 0 \\ 0 & 0 & 1-i & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1-i}{4} & \frac{1-i}{2} & 0 \\ 0 & 1 & -i & \frac{1-i}{4} & \frac{-1-i}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1+i}{2} \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1-i}{4} & \frac{1-i}{2} & \frac{-1-i}{2} \\ 0 & 1 & 0 & \frac{1-i}{4} & \frac{-1-i}{2} & \frac{-1+i}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1+i}{2} \end{array} \right] \end{aligned}$$

Therefore

$$P^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1-i}{4} & \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{1-i}{4} & \frac{-1-i}{2} & \frac{-1+i}{2} \\ 0 & 0 & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1-3i}{4} \\ \frac{-1+i}{4} \\ \frac{1+i}{2} \end{bmatrix}$$

Thus $(1, 0, 1) = \frac{-1-3i}{4}(2i, 1, 0) + \frac{-1+i}{4}(2, -1, 0) + \frac{1+i}{2}(0, 1+i, 1-i)$.

Exercise 3: Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for \mathbb{R}^3 consisting of

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0).$$

What are the coordinates of the vector (a, b, c) in the ordered basis \mathcal{B} ?

Solution: Using Theorem 7, page 52, the answer is $P^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

We find P^{-1} by row-reducing the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right].$$

The right side will transform into the P^{-1} . Row reducing:

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array} \right]. \end{aligned}$$

Therefore,

$$P^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b - c \\ b \\ a - 2b + c \end{bmatrix}$$

Thus the answer is

$$[(a, b, c)]_{\mathcal{B}} = (b - c, b, a - 2b + c).$$

Exercise 4: Let W be the subspace of \mathbb{C}^3 spanned by $\alpha_1 = (1, 0, i)$ and $\alpha_2 = (1 + i, 1, -1)$.

- Show that α_1 and α_2 form a basis for W .
- Show that the vectors $\beta_1 = (1, 1, 0)$ and $\beta_2 = (1, i, 1 + i)$ are in W and form another basis for W .
- What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for W ?

Solution: (a) To show α_1 and α_2 form a basis of the space they generate we must show they are linearly independent. In other words that $a\alpha_1 + b\alpha_2 = 0 \Rightarrow a = b = 0$. Equivalently we need to show neither is a multiple of the other. If $\alpha_2 = c\alpha_1$ then from the second component it follows that $c = 0$ which would imply $\alpha_2 = (0, 0, 0)$, which it does not. So $\{\alpha_1, \alpha_2\}$ is a basis for the space they generate.

(b) Since the first coordinate of both β_1 and β_2 is one, it's clear that neither is a multiple of the other. So they generate a two dimensional subspace of \mathbb{C}^3 . If we show β_1 and β_2 can be written as linear combinations of α_1 and α_2 then since the spaces generated by them both have dimension two, by Corollary 1, page 46, they must be equal. To show β_1 and β_2 can be written as linear combinations of α_1 and α_2 we row-reduce the augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 1+i & 1 & 1 \\ 0 & 1 & 1 & i \\ i & -1 & 0 & 1+i \end{array} \right].$$

Row reduction follows:

$$\left[\begin{array}{cc|cc} 1 & 1+i & 1 & 1 \\ 0 & 1 & 1 & i \\ i & -1 & 0 & 1+i \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 1 \\ 0 & 1 & 1 & i \\ 0 & -i & -i & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -i & 2-i \\ 0 & 1 & 1 & i \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Thus $\beta_1 = -i\alpha_1 + \alpha_2$ and $\beta_2 = (2 - i)\alpha_1 + i\alpha_2$.

(c) We have to write the β_i 's in terms of the α_i 's, basically the opposite of what we did in part b. In this case we row-reduce the augmented matrix

$$\begin{aligned} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1+i \\ 1 & i & 0 & 1 \\ 0 & 1+i & i & -1 \end{array} \right] &\rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1+i \\ 0 & -1+i & -1 & -i \\ 0 & 1+i & i & -1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1+i \\ 0 & -1+i & -1 & -i \\ 0 & 0 & 0 & 0 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1+i \\ 0 & 1 & \frac{1+i}{2} & \frac{-1+i}{2} \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1-i}{2} & \frac{3+i}{2} \\ 0 & 1 & \frac{1+i}{2} & \frac{-1+i}{2} \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

Thus $\alpha_1 = \frac{1-i}{2}\beta_1 + \frac{1+i}{2}\beta_2$ and $\alpha_2 = \frac{3+i}{2}\beta_1 + \frac{-1+i}{2}\beta_2$. So finally, if \mathcal{B} is the basis $\{\beta_1, \beta_2\}$ then

$$[\alpha_1]_{\mathcal{B}} = \left(\frac{1-i}{2}, \frac{1+i}{2} \right)$$

$$[\alpha_2]_{\mathcal{B}} = \left(\frac{3+i}{2}, \frac{-1+i}{2} \right).$$

Exercise 5: Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ be vectors in \mathbb{R}^2 such that

$$x_1y_1 + x_2y_2 = 0, \quad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1.$$

Prove that $\mathcal{B} = \{\alpha, \beta\}$ is a basis for \mathbb{R}^2 . Find the coordinates of the vector (a, b) in the ordered basis $\mathcal{B} = \{\alpha, \beta\}$. (The conditions on α and β say, geometrically, that α and β are perpendicular and each has length 1.)

Solution: It suffices by Corollary 1, page 46, to show α and β are linearly independent, because then they generate a subspace of \mathbb{R}^2 of dimension two, which therefore must be all of \mathbb{R}^2 . The second condition on x_1, x_2, y_1, y_2 implies that neither α nor β are the zero vector. To show two vectors are linearly independent we only need show neither is a non-zero scalar multiple of the other. Suppose WLOG that $\beta = c\alpha$ for some $c \in \mathbb{R}$, and since neither vector is the zero vector, $c \neq 0$. Then $y_1 = cx_1$ and $y_2 = cx_2$. Thus the conditions on x_1, x_2, y_1, y_2 implies

$$0 = x_1y_1 + x_2y_2 = cx_1^2 + cx_2^2 = c(x_1^2 + x_2^2) = c \cdot 1 = c.$$

Thus $c = 0$, a contradiction.

It remains to find the coordinates of the arbitrary vector (a, b) in the ordered basis $\{\alpha, \beta\}$. To find the coordinates of (a, b) we can row-reduce the augmented matrix

$$\left[\begin{array}{cc|c} x_1 & y_1 & a \\ x_2 & y_2 & b \end{array} \right].$$

It cannot be that both $x_1 = x_2 = 0$ so assume WLOG that $x_1 \neq 0$. Also it cannot be that both $y_1 = y_2 = 0$. Assume first that $y_1 \neq 0$. Since order matters we cannot assume $y_1 \neq 0$ WLOG, so we must consider both cases. Then note that $x_1y_1 + x_2y_2 = 0 \Rightarrow$

$$\frac{x_2y_2}{x_1y_1} = -1 \tag{18}$$

Thus if $x_1y_2 - x_2y_1 = 0$ then $\frac{x_2}{x_1} = \frac{y_2}{y_1}$ from which (18) implies $\left(\frac{x_2}{x_1}\right)^2 = -1$, a contradiction. Thus we can conclude that $x_1y_2 - x_2y_1 \neq 0$. We use this in the following row reduction to be sure we are not dividing by zero.

$$\left[\begin{array}{cc|c} x_1 & y_1 & a \\ x_2 & y_2 & b \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & y_1/x_1 & a/x_1 \\ x_2 & y_2 & b \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & y_1/x_1 & a/x_1 \\ 0 & y_2 - \frac{x_2y_1}{x_1} & b - \frac{x_2a}{x_1} \end{array} \right] = \left[\begin{array}{cc|c} 1 & y_1/x_1 & a/x_1 \\ 0 & \frac{x_1y_2 - x_2y_1}{x_1} & \frac{bx_1 - ax_2}{x_1} \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & y_1/x_1 & a/x_1 \\ 0 & 1 & \frac{bx_1 - ax_2}{x_1 y_2 - x_2 y_1} \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{ay_2 - by_1}{x_1 y_2 - x_2 y_1} \\ 0 & 1 & \frac{bx_1 - ax_2}{x_1 y_2 - x_2 y_1} \end{array} \right]$$

Now if we substitute $y_1 = -x_2 y_2 / x_1$ into the numerator and denominator of $\frac{ay_2 - by_1}{x_1 y_2 - x_2 y_1}$ and use $x_1^2 + x_2^2 = 1$ it simplifies to $ax_1 + bx_2$. Similarly $\frac{bx_1 - ax_2}{x_1 y_2 - x_2 y_1}$ simplifies to $ay_1 + by_2$. So we get

$$\left[\begin{array}{cc|c} 1 & 0 & ax_1 + bx_2 \\ 0 & 1 & ay_1 + by_2 \end{array} \right].$$

Now assume $y_2 \neq 0$ (and we continue to assume $x_1 \neq 0$ since we assumed that WLOG). In this case

$$\frac{y_1}{y_2} = -\frac{x_2}{x_1} \quad (19)$$

So if $x_1 y_2 - x_2 y_1 = 0$ then $\frac{x_2 y_1}{x_1 y_2} = 1$. But then (19) implies $\left(\frac{x_2}{x_1}\right)^2 = -1$ a contradiction. So also in this case we can assume $x_1 y_2 - x_2 y_1 \neq 0$ and so we can do the same row-reduction as before. Thus in all cases

$$(ax_1 + bx_2)\alpha + (ay_1 + by_2)\beta = (a, b)$$

or equivalently

$$(ax_1 + bx_2)(x_1, x_2) + (ay_1 + by_2)(y_1, y_2) = (a, b).$$

Exercise 6: Let V be the vector space over the complex numbers of all functions from \mathbb{R} into \mathbb{C} , i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

- (a) Prove that f_1 , f_2 , and f_3 are linearly independent.
 (b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible 3×3 matrix P such that

$$g_j = \sum_{i=1}^3 P_{ij} f_i.$$

Solution: Suppose $a + be^{ix} + ce^{-ix} = 0$. Let $y = e^{ix}$. Then $y \neq 0$ and $a + by + \frac{c}{y} = 0$ which implies $ay + by^2 + c = 0$. This is a quadratic polynomial in y thus can be zero for at most two values of y . But e^{ix} is a continuous non-constant function, so it cannot be zero for all $y = e^{ix}$, so this is a contradiction.

We know that $e^{ix} = \cos(x) + i \sin(x)$. Thus $e^{-ix} = \cos(x) - i \sin(x)$. Adding these gives $2 \cos(x) = e^{ix} + e^{-ix}$. Thus $\cos(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$. Subtracting instead of adding the equations gives $e^{ix} - e^{-ix} = 2i \sin(x)$. Thus $\sin(x) = \frac{1}{2i}e^{ix} + \frac{1}{2i}e^{-ix}$ or equivalently $\sin(x) = -\frac{i}{2}e^{ix} - \frac{i}{2}e^{-ix}$. Thus the requested matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & 1/2 & -i/2 \end{bmatrix}.$$

Exercise 7: Let V be the (real) vector space of all polynomial functions from \mathbb{R} into \mathbb{R} of degree 2 or less, i.e., the space of all functions f of the form

$$f(x) = c_0 + c_1 x + c_2 x^2.$$

Let t be a fixed real number and define

$$g_1(x) = 1, \quad g_2(x) = x + t, \quad g_3(x) = (x + t)^2.$$

Prove that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V . If

$$f(x) = c_0 + c_1 x + c_2 x^2$$

what are the coordinates of f in this ordered basis \mathcal{B} ?

Solution: We know V has dimension three (it follows from Example 16, page 43, that $\{1, x, x^2\}$ is a basis). Thus by Corollary 2 (b), page 45, it suffices to show $\{g_1, g_2, g_3\}$ span V . We need to solve for u, v, w the equation

$$c_2x^2 + c_1x + c_0 = u + v(x+t) + w(x+t)^2.$$

Rearranging

$$c_2x^2 + c_1x + c_0 = wx^2 + (v + 2wt)x + (u + vt + wt^2).$$

It follows that

$$w = c_2$$

$$v = c_1 - 2c_2t$$

$$u = c_0 - c_1t + c_2t^2.$$

Thus $\{g_1, g_2, g_3\}$ do span V and the coordinates of $f(x) = c_2x^2 + c_1x + c_0$ are

$$(c_2, c_1 - 2c_2t, c_0 - c_1t + c_2t^2).$$

Section 2.6: Computations Concerning Subspaces

Exercise 1: Let $s < n$ and A and $s \times n$ matrix with entries in the field F . Use Theorem 4 (not its proof) to show that there is a non-zero X in $F^{n \times 1}$ such that $AX = 0$.

Solution: Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the columns of A . Then $\alpha_i \in F^s \forall i$. Thus $\{\alpha_1, \dots, \alpha_n\}$ are n vectors in F^s . But F^s has dimension $s < n$ thus by Theorem 4, page 44, $\alpha_1, \dots, \alpha_n$ cannot be linearly independent. Thus $\exists x_1, \dots, x_n$ such that $x_1\alpha_1 + \dots + x_n\alpha_n = 0$. Thus if

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

then $AX = x_1\alpha_1 + \dots + x_n\alpha_n = 0$.

Exercise 2: Let

$$\alpha_1 = (1, 1, -2, 1), \quad \alpha_2 = (3, 0, 4, -1), \quad \alpha_3 = (-1, 2, 5, 2).$$

Let

$$\alpha = (4, -5, 9, -7), \quad \beta = (3, 1, -4, 4), \quad \gamma = (-1, 1, 0, 1).$$

- Which of the vectors α, β, γ are in the subspace of \mathbb{R}^4 spanned by the α_i ?
- Which of the vectors α, β, γ are in the subspace of \mathbb{C}^4 spanned by the α_i ?
- Does this suggest a theorem?

Solution:

(a) We use the approach of row-reducing the matrix whose rows are given by the α_i :

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 3 & 0 & 4 & -1 \\ -1 & 2 & 5 & 2 \end{bmatrix}$$

$$\begin{aligned} &\rightarrow \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -3 & 10 & -4 \\ 0 & 3 & 3 & 3 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 0 & 13 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 13 & -1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1/13 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 0 & -3/13 \\ 0 & 1 & 0 & 14/13 \\ 0 & 0 & 1 & -1/13 \end{bmatrix} \end{aligned}$$

Let $\rho_1 = (1, 0, 0, -3/13)$, $\rho_2 = (0, 1, 0, 14/13)$ and $\rho_3 = (0, 0, 1, -1/13)$. Thus elements of the subspace spanned by the α_i are of the form $b_1\rho_1 + b_2\rho_2 + b_3\rho_3$

$$= (b_1, b_2, b_3, \frac{1}{13}(14b_2 - 3b_1 - b_3)).$$

- $\alpha = (4, -5, 9, -7)$. We have $b_1 = 4$, $b_2 = -5$ and $b_3 = 9$. Thus if α is in the subspace it must be that

$$\frac{1}{13}(14(-5) - 3(4) - 9) \stackrel{?}{=} b_4$$

where $b_4 = 7$. Indeed the left hand side does equal 7, so α is in the subspace.

- $\beta = (3, 1, -4, 4)$. We have $b_1 = 3$, $b_2 = 1$, $b_3 = -4$. Thus if β is in the subspace it must be that

$$\frac{1}{13}(14 - 3(3) + 4) \stackrel{?}{=} b_4$$

where $b_4 = 4$. But the left hand side equals $9/13 \neq 4$ so β is *not* in the subspace.

- $\gamma = (-1, 1, 0, 1)$. We have $b_1 = -1$, $b_2 = 1$, $b_3 = 0$. Thus if γ is in the subspace it must be that

$$\frac{1}{13}(14 - 3(-1) - 0) \stackrel{?}{=} b_4$$

where $b_4 = 1$. But the left hand side equals $17/13 \neq 1$ so γ is *not* in the subspace.

(b) Nowhere in the above did we use the fact that the field was \mathbb{R} instead of \mathbb{C} . The only equations we had to solve are linear equations with real coefficients, which have solutions in \mathbb{R} if and only if they have solutions in \mathbb{C} . Thus the same results hold: α is in the subspace while β and γ are not.

(c) This suggests the following theorem: Suppose F is a subfield of the field E and $\alpha_1, \dots, \alpha_n$ are a basis for a subspace of F^n , and $\alpha \in F^n$. Then α is in the subspace of F^n generated by $\alpha_1, \dots, \alpha_n$ if and only if α is in the subspace of E^n generated by $\alpha_1, \dots, \alpha_n$.

Exercise 3: Consider the vectors in \mathbb{R}^4 defined by

$$\alpha_1 = (-1, 0, 1, 2), \quad \alpha_2 = (3, 4, -2, 5), \quad \alpha_3 = (1, 4, 0, 9).$$

Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of \mathbb{R}^4 spanned by the three given vectors.

Solution: We use the approach of row-reducing the matrix whose rows are given by the α_i :

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 4 & 1 & 11 \\ 0 & 4 & 1 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1/4 & 11/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\rho_1 = (1, 0, -1, -2)$ and $\rho_2 = (0, 1, 1/4, 11/4)$. Then the arbitrary element of the subspace spanned by α_1 and α_2 is of the form $b_1\rho_1 + b_2\rho_2$ for arbitrary $b_1, b_2 \in \mathbb{R}$. Expanding we get

$$b_1\rho_1 + b_2\rho_2 = (b_1, b_2, -b_1 + \frac{1}{4}b_2, -2b_2 + \frac{11}{4}b_2).$$

Thus the equations that must be satisfied for (x, y, z, w) to be in the subspace are

$$\begin{cases} z = -x + \frac{1}{4}y \\ w = -2x + \frac{11}{4}y \end{cases}.$$

or equivalently

$$\begin{cases} -x + \frac{1}{4}y - z = 0 \\ -2x + \frac{11}{4}y - w = 0 \end{cases}.$$

Exercise 4: In \mathbb{C}^3 let

$$\alpha_1 = (1, 0, -i), \quad \alpha_2 = (1 + i, 1 - i, 1), \quad \alpha_3 = (i, i, i).$$

Prove that these vectors form a basis for \mathbb{C}^3 . What are the coordinates of the vector (a, b, c) in this basis?

Solution: We use the approach of row-reducing the augmented matrix:

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 0 & -i & 1 & 0 & 0 \\ 1+i & 1-i & 1 & 0 & 1 & 0 \\ i & i & i & 0 & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -i & 1 & 0 & 0 \\ 0 & 1-i & i & -1-i & 1 & 0 \\ 0 & i & i-1 & -i & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & \frac{-1+i}{2} & -i & \frac{1+i}{2} & 0 \\ 0 & i & i-1 & -i & 0 & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & \frac{-1+i}{2} & -i & \frac{1+i}{2} & 0 \\ 0 & 0 & \frac{-1+3i}{2} & -1-i & \frac{1-i}{2} & 1 \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & \frac{-1+i}{2} & -i & \frac{1+i}{2} & 0 \\ 0 & 0 & 1 & \frac{-2+4i}{5} & \frac{-2-i}{5} & \frac{-1-3i}{5} \end{array} \right] \\ & \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1-2i}{5} & \frac{1-2i}{5} & \frac{3-i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2-i}{5} \\ 0 & 0 & 1 & \frac{-2+4i}{5} & \frac{-2-i}{5} & \frac{-1-3i}{5} \end{array} \right] \end{aligned}$$

Since the left side transformed into the identity matrix we know that $\{\alpha_1, \alpha_2, \alpha_3\}$ form a basis for \mathbb{C}^3 . The matrix on the right is P^{-1} from (2-17), so the coordinate matrix of (a, b, c) with respect to the basis $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ are given by

$$\begin{aligned} [(a, b, c)]_{\mathcal{B}} &= P^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} \frac{1-2i}{5}a + \frac{1-2i}{5}b + \frac{3-i}{5}c \\ \frac{1-2i}{5}a + \frac{1+3i}{5}b + \frac{-2-i}{5}c \\ \frac{-2+4i}{5}a + \frac{-2-i}{5}b + \frac{-1-3i}{5}c \end{bmatrix}. \end{aligned}$$

Exercise 5: Give an explicit description of the type (2-25) for the vectors

$$\beta = (b_1, b_2, b_3, b_4, b_5)$$

in \mathbb{R}^5 which are linear combinations of the vectors

$$\alpha_1 = (1, 0, 2, 1, -1), \quad \alpha_2 = (-1, 2, -4, 2, 0)$$

$$\alpha_3 = (2, -1, 5, 2, 1), \quad \alpha_4 = (2, 1, 3, 5, 2).$$

Solution: We row-reduce the matrix whose rows are given by the α_i 's.

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ -1 & 2 & -4 & 2 & 0 \\ 2 & -1 & 5 & 2 & 1 \\ 2 & 1 & 3 & 5 & 2 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 2 & -2 & 3 & -1 \\ 0 & -1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 3 & 4 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 3 & 7 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 & 7 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 & -4 \\ 0 & 1 & -1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & -1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Let $\rho_1 = (1, 0, 2, 0, 0)$, $\rho_2 = (0, 1, -1, 0, 0)$, $\rho_3 = (0, 0, 0, 1, 0)$ and $\rho_4 = (0, 0, 0, 0, 1)$. Then the general element that is a linear combination of the ρ_i 's is $b_1\rho_1 + b_2\rho_2 + b_3\rho_3 + b_4\rho_4 = (b_1, b_2, 2b_1 - b_2, b_3, b_4)$.

Exercise 6: Let V be the real vector space spanned by the rows of the matrix

$$A = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}.$$

- (a) Find a basis for V .
- (b) Tell which vectors $(x_1, x_2, x_3, x_4, x_5)$ are elements of V .
- (c) If $(x_1, x_2, x_3, x_4, x_5)$ is in V what are its coordinates in the basis chosen in part (a)?

Solution: We row-reduce the augmented matrix

$$\begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 2 & 10 & 3 \\ 0 & 0 & 5 & 25 & 6 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 2 & 10 & 3 \\ 0 & 0 & 5 & 25 & 6 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) A basis for V is given by the non-zero rows of the reduced matrix

$$\rho_1 = (1, 7, 0, 3, 0), \quad \rho_2 = (0, 0, 1, 5, 0), \quad \rho_3 = (0, 0, 0, 0, 1).$$

(b) Vectors of V are any of the form $b_1\rho_1 + b_2\rho_2 + b_3\rho_3$

$$= (b_1, 7b_1, b_2, 3b_1 + 5b_2, b_3)$$

for arbitrary $b_1, b_2, b_3 \in \mathbb{R}$.

(c) By the above, the element $(x_1, x_2, x_3, x_4, x_5)$ in V must be of the form $x_1\rho_1, x_3\rho_2, x_5\rho_3$. In other words if $\mathcal{B} = \{\rho_1, \rho_2, \rho_3\}$ is the basis for V given in part (a), then the coordinate matrix of $(x_1, x_2, x_3, x_4, x_5)$ is

$$[(x_1, x_2, x_3, x_4, x_5)]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix}.$$

Exercise 7: Let A be an $m \times n$ matrix over the field F , and consider the system of equations $AX = Y$. Prove that this system of equations has a solution if and only if the row rank of A is equal to the row rank of the augmented matrix of the system.

Solution: To solve the system we row-reduce the augmented matrix $[A|Y]$ resulting in an augmented matrix $[R|Z]$ where R is in reduced echelon form and Z is an $m \times 1$ matrix. The system has a solution if and only if whenever the last k rows of R are zero rows then the last r entries of Z are also zeros. If this were not the case then we would have $\alpha = 0$ for a non-zero α . Thus the only non-zero entries in Z are in subset of the rows for which R is non-zero. In this case the rank of the augmented matrix is clearly the same as the rank of R . Thus the system has a solution if and only if the rank of R is the same as the rank of $[R|Z]$ which has the same rank as $[A|Y]$ since they differ by elementary row operations.

Chapter 3: Linear Transformations

Section 3.1: Linear Transformations

Exercise 1: Which of the following functions T from \mathbb{R}^2 into \mathbb{R}^2 are linear transformations?

- (a) $T(x_1, x_2) = (1 + x_1, x_2)$;
- (b) $T(x_1, x_2) = (x_2, x_1)$;
- (c) $T(x_1, x_2) = (x_1^2, x_2)$;
- (d) $T(x_1, x_2) = (\sin x_1, x_2)$;
- (e) $T(x_1, x_2) = (x_1 - x_2, 0)$.

Solution:

(a) T is not a linear transformation because $T(0, 0) = (1, 0)$ and according to the comments after Example 5 on page 68 we know that it must always be that $T(0, 0) = (0, 0)$.

(b) T is a linear transformation. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$. Then $T(c\alpha + \beta) = T((cx_1 + y_1, cx_2 + y_2)) = (cx_2 + y_2, cx_1 + y_1) = c(x_2, x_1) + (y_2, y_1) = cT(\alpha) + T(\beta)$.

(c) T is not a linear transformation. If T were a linear transformation then we'd have $(1, 0) = T((-1, 0)) = T(-1 \cdot (1, 0)) = -1 \cdot T(1, 0) = -1 \cdot (1, 0) = (-1, 0)$ which is a contradiction, $(1, 0) \neq (-1, 0)$.

(d) T is not a linear transformation. If T were a linear transformation then $(0, 0) = T(\pi, 0) = T(2(\pi/2, 0)) = 2T((\pi/2, 0)) = 2(\sin(\pi/2), 0) = 2(1, 0) = (2, 0)$ which is a contradiction, $(0, 0) \neq (2, 0)$.

(e) T is a linear transformation. Let $Q = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Then (identifying \mathbb{R}^2 with $\mathbb{R}^{1 \times 2}$) $T(x_1, x_2) = [x_1 \ x_2]Q$ so from Example 4, page 68, (with P being the identity matrix), it follows that T is a linear transformation.

Exercise 2: Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional vector space V .

Solution: Suppose V has dimension n . The range of the zero transformation is the zero subspace $\{0\}$; the range of the identity transformation is the whole space V . The rank of the zero transformation is the dimension of the range which is zero; the rank of the identity transformation is the rank of the whole space V which is n . The null space of the zero transformation is the whole space V ; the null space of the identity transformation is the zero subspace $\{0\}$. The nullity of the zero transformation is the dimension of its null space, which is the whole space, so is n ; the nullity of the identity transformation is the dimension of its null space, which is the zero space, so is 0.

Exercise 3: Describe the range and the null space for the differentiation transformation of Example 2. Do the same for the integration transformation of Example 5.

Solution: V is the space of polynomials. The range of the differentiation transformation is all of V since if $f(x) = c_0 + c_1x + \cdots + c_nx^n$ then $f(x) = (Dg)(x)$ where $g(x) = c_0x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \cdots + \frac{c_n}{n+1}x^{n+1}$. The null space of the differentiation transformation is the set of constant polynomials since $(Dc)(x) = 0$ for constants $c \in F$.

The range of the integration transformation is all polynomials with constant term equal to zero. Let $f(x) = c_1x + c_2x^2 + \cdots + c_nx^n$. Then $f(x) = (Tg)(x)$ where $g(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1}$. Clearly the integral transformation of a polynomial has constant term equal to zero, so this is the entire range of the integration transformation. The null space of the integration transformation is the zero space $\{0\}$ since the integral of any other polynomial is non-zero.

Exercise 4: Is there a linear transformation T from \mathbb{R}^3 into \mathbb{R}^2 such that $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$?

Solution: Yes, there is such a linear transformation. Clearly $\alpha_1 = (1, -1, 1)$ and $\alpha_2 = (1, 1, 1)$ are linearly independent. By Corollary 2, page 46, \exists a third vector α_3 such that $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis for \mathbb{R}^3 . By Theorem 1, page 69, there is a linear transformation that takes $\alpha_1, \alpha_2, \alpha_3$ to any three vectors we want. Therefore we can find a linear transformation that takes $\alpha_1 \mapsto (1, 0)$, $\alpha_2 \mapsto (0, 1)$ and $\alpha_3 \mapsto (0, 0)$. (We could have used any vector instead of $(0, 0)$.)

Exercise 5: If

$$\alpha_1 = (1, -1), \quad \beta_1 = (1, 0)$$

$$\alpha_2 = (2, -1), \quad \beta_2 = (0, 1)$$

$$\alpha_3 = (-3, 2), \quad \beta_3 = (1, 1)$$

is there a linear transformation T from \mathbb{R}^2 to \mathbb{R}^2 such that $T\alpha_i = \beta_i$ for $i = 1, 2$ and 3 ?

Solution: No there is no such transformation. If there was then since $\{\alpha_1, \alpha_2\}$ is a basis for \mathbb{R}^2 their images determine T completely. Now $\alpha_3 = -\alpha_1 - \alpha_2$, thus it must be that $T(\alpha_3) = T(-\alpha_1 - \alpha_2) = -T(\alpha_1) - T(\alpha_2) = -(1, 0) - (0, 1) = (-1, -1) \neq (1, 1)$. Thus no such T can exist.

Exercise 6: Describe explicitly (as in Exercises 1 and 2) the linear transformation T from F^2 into F^2 such that $T\epsilon_1 = (a, b)$, $T\epsilon_2 = (c, d)$.

Solution: I'm not 100% sure I understand what they want here. Let A be the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the range of T is the row-space of A which can have dimension 0, 1, or 2 depending on the row-rank. Explicitly it is all vectors of the form $x(a, b) + y(c, d) = (ax + cy, bx + dy)$ where x, y are arbitrary elements of F . The rank is the dimension of this row-space, which is 0 if $a = b = c = d = 0$ and if not all a, b, c, d are zero then by Exercise 1.6.8, page 27, the rank is 2 if $ad - bc \neq 0$ and equals 1 if $ad - bc = 0$.

Now let A be the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then the null space is the solution space of $AX = 0$. Thus the nullity is 2 if $a = b = c = d = 0$, and if not all a, b, c, d are zero then by Exercise 1.6.8, page 27 and Theorem 13, page 23, is 0 if $ad - bc \neq 0$ and is 1 if $ad - bc = 0$.

Exercise 7: Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3).$$

(a) Verify that T is a linear transformation.

(b) If (a, b, c) is a vector in F^3 , what are the conditions on a, b , and c that the vector be in the range of T ? What is the rank of T ?

(c) What are the conditions on a, b , and c that (a, b, c) be in the null space of T ? What is the nullity of T ?

Solution: (a) Let

$$P = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}.$$

Then T can be represented by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} \mapsto P \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}.$$

By Example 4, page 68, this is a linear transformation, where we've identified F^3 with $F^{3 \times 1}$ and taken Q in Example 4 to be the identity matrix.

(b) The range of T is the row space of the matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix}.$$

We row reduce the matrix as follows

$$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix}.$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\rho_1 = (1, 0, 1)$ and $\rho_2 = (0, 1, -1)$. Then elements of the row space are elements of the form $b_1\rho_1 + b_2\rho_2 = (b_1, b_2, b_1 - b_2)$. Thus the rank of T is two and (a, b, c) is in the range of T as long as $c = a - b$.

Alternatively, we can row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 2 & 1 & 0 & b \\ -1 & -2 & 2 & c \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 3 & -4 & b-2a \\ 0 & -3 & 4 & a+c \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 3 & -4 & b-2a \\ 0 & 0 & 0 & -a+b+c \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 1 & -4/3 & (b-2a)/4 \\ 0 & 0 & 0 & -a+b+c \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & (b+2a)/4 \\ 0 & 1 & -4/3 & (b-2a)/4 \\ 0 & 0 & 0 & -a+b+c \end{array} \right]$$

from which we arrive at the condition $-a + b + c = 0$ or equivalently $c = a - b$.

(c) We must find all $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $PX = 0$ where P is the matrix from part (a). We row reduce the matrix

$$\begin{aligned} & \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \\ & \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix} \\ & \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \\ & \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix} \\ & \rightarrow \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Therefore

$$\begin{cases} a + \frac{2}{3}c = 0 \\ b - \frac{4}{3}c = 0 \end{cases}$$

So elements of the null space of T are of the form $(-\frac{2}{3}c, \frac{4}{3}c, c)$ for arbitrary $c \in F$ and the dimension of the null space (the nullity) equals one.

Exercise 8: Describe explicitly a linear transformation from \mathbb{R}^3 to \mathbb{R}^3 which has as its range the subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$.

Solution: By Theorem 1, page 69, (and its proof) there is a linear transformation T from \mathbb{R}^3 to \mathbb{R}^3 such that $T(1, 0, 0) = (1, 0, -1)$, $T(0, 1, 0) = (1, 0, -1)$ and $T(0, 0, 1) = (1, 2, 2)$ and the range of T is exactly the subspace generated by $\{T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)\} = \{(1, 0, -1), (1, 2, 2)\}$.

Exercise 9: Let V be the vector space of all $n \times n$ matrices over the field F , and let B be a fixed $n \times n$ matrix. If

$$T(A) = AB - BA$$

verify that T is a linear transformation from V into V .

Solution: $T(cA_1 + A_2) = (cA_1 + A_2)B - B(cA_1 + A_2) = cA_1B + A_2B + cBA_1 + BA_2$
 $= c(A_1B + BA_1) + (A_2B + BA_2) = cT(A_1) + T(A_2)$.

Exercise 10: Let V be the set of all complex numbers regarded as a vector space over the field of real numbers (usual operations). Find a function from V into V which is a linear transformation on the above vector space, but which is not a linear

transformation on \mathbb{C}^1 , i.e., which is not complex linear.

Solution: Let $T : V \rightarrow V$ be given by $a + bi \mapsto a$. Let $z = a + bi$ and $w = a' + b'i$ and $c \in \mathbb{R}$. Then $T(cz + w) = T((ca + a') + (cb + b')i) = ca + a' = cT(a + bi) + T(a' + b'i) = aT(z) + T(w)$. Thus T is real linear. However, if T were complex linear then we must have $0 = T(i) = T(i \cdot 1) = i \cdot T(1) = i \cdot 1 = i$. But $0 \neq i$ so this is a contradiction. Thus T is not complex linear.

Exercise 11: Let V be the space of $n \times 1$ matrices over F and let W be the space of $m \times 1$ matrices over F . Let A be a fixed $m \times n$ matrix over F and let T be the linear transformation from V into W defined by $T(X) = AX$. Prove that T is the zero transformation if and only if A is the zero matrix.

Solution: If A is the zero matrix then clearly T is the zero transformation. Conversely, suppose A is not the zero matrix, suppose the k -th column A_k has a non-zero entry. Then $T(\epsilon_k) = A_k \neq 0$.

Exercise 12: Let V be an n -dimensional vector space over the field F and let T be a linear transformation from V into V such that the range and null space of T are identical. Prove that n is even. (Can you give an example of such a linear transformation T ?)

Solution: From Theorem 2, page 71, we know $\text{rank}(T) + \text{nullity}(T) = \dim V$. In this case we are assuming both terms on the left hand side are equal, say equal to m . Thus $m + m = n$ or equivalently $n = 2m$ which implies n is even.

The simplest example is $V = \{0\}$ the zero space. Then trivially the range and null space are equal. To give a less trivial example assume $V = \mathbb{R}^2$ and define T by $T(1, 0) = (0, 0)$ and $T(0, 1) = (1, 0)$. We can do this by Theorem 1, page 69 because $\{(1, 0), (0, 1)\}$ is a basis for \mathbb{R}^2 . Then clearly the range and null space are both equal to the subspace of \mathbb{R}^2 generated by $(1, 0)$.

Exercise 13: Let V be a vector space and T a linear transformation from V into V . Prove that the following two statements about T are equivalent.

- (a) The intersection of the range of T and the null space of T is the zero subspace of V .
- (b) If $T(T\alpha) = 0$, then $T\alpha = 0$.

Solution: (a) \Rightarrow (b): Statement (a) says that nothing in the range gets mapped to zero except for 0. In other words if x is in the range of T then $Tx = 0 \Rightarrow x = 0$. Now $T\alpha$ is in the range of T , thus $T(T\alpha) = 0 \Rightarrow T\alpha = 0$.

(b) \Rightarrow (a): Suppose x is in both the range and null space of T . Since x is in the range, $x = T\alpha$ for some α . But then x in the null space of T implies $T(x) = 0$ which implies $T(T\alpha) = 0$. Thus statement (b) implies $T\alpha = 0$ or equivalently $x = 0$. Thus the only thing in both the range and null space of T is the zero vector 0.

Section 3.2: The Algebra of Linear Transformations

Page 76: Typo in line 1: It says A_{ij}, \dots, A_{mj} , it should say A_{1j}, \dots, A_{mj} .

Exercise 1: Let T and U be the linear operators on \mathbb{R}^2 defined by

$$T(x_1, x_2) = (x_2, x_1) \quad \text{and} \quad U(x_1, x_2) = (x_1, 0).$$

- (a) How would you describe T and U geometrically?
- (b) Give rules like the ones defining T and U for each of the transformations $(U + T)$, UT , TU , T^2 , U^2 .

Solution: (a) Geometrically, in the x - y plane, T is the reflection about the diagonal $x = y$ and U is a projection onto the x -axis.

(b)

- $(U + T)(x_1, x_2) = (x_2, x_1) + (x_1, 0) = (x_1 + x_2, x_1)$.
- $(UT)(x_1, x_2) = U(x_2, x_1) = (x_2, 0)$.
- $(TU)(x_1, x_2) = T(x_1, 0) = (0, x_1)$.
- $T^2(x_1, x_2) = T(x_2, x_1) = (x_1, x_2)$, the identity function.
- $U^2(x_1, x_2) = U(x_1, 0) = (x_1, 0)$. So $U^2 = U$.

Exercise 2: Let T be the (unique) linear operator on \mathbb{C}^3 for which

$$T\epsilon_1 = (1, 0, i), \quad T\epsilon_2 = (0, 1, 1), \quad T\epsilon_3 = (i, 1, 0).$$

Is T invertible?

Solution: By Theorem 9 part (iv), top of page 82, T is invertible if $\{T\epsilon_1, T\epsilon_2, T\epsilon_3\}$ is a basis of \mathbb{C}^3 . Since \mathbb{C}^3 has dimension three, it suffices (by Corollary 1 page 46) to show $T\epsilon_1, T\epsilon_2, T\epsilon_3$ are linearly independent. To do this we row reduce the matrix

$$\begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{bmatrix}$$

to row-reduced echelon form. If it reduces to the identity then its rows are independent, otherwise they are dependent. Row reduction follows:

$$\begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is in row-reduced echelon form not equal to the identity. Thus T is not invertible.

Exercise 3: Let T be the linear operator on \mathbb{R}^3 defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

Is T invertible? If so, find a rule for T^{-1} like the one which defines T .

Solution: The matrix representation of the transformation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where we've identified \mathbb{R}^3 with $\mathbb{R}^{3 \times 1}$. T is invertible if the matrix of the transformation is invertible. To determine this we row-reduce the matrix - we row-reduce the augmented matrix to determine the inverse for the second part of the Exercise.

$$\begin{bmatrix} 3 & 0 & 0 & | & 1 & 0 & 0 \\ 1 & -1 & 0 & | & 0 & 1 & 0 \\ 2 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \\ \rightarrow \begin{bmatrix} 1 & -1 & 0 & | & 0 & 1 & 0 \\ 3 & 0 & 0 & | & 1 & 0 & 0 \\ 2 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} &\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & -3 & 0 \\ 0 & 3 & 1 & 0 & -2 & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/3 & -1 & 0 \\ 0 & 3 & 1 & 0 & -2 & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 1/3 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array} \right] \end{aligned}$$

Since the left side transformed into the identity, T is invertible. The inverse transformation is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So

$$T^{-1}(x_1, x_2, x_3) = (x_1/3, x_1/3 - x_2, -x_1 + x_2 + x_3).$$

Exercise 4: For the linear operator T of Exercise 3, prove that

$$(T^2 - I)(T - 3I) = 0.$$

Solution: Working with the matrix representation of T we must show

$$(A^2 - I)(A - 3I) = 0$$

where

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

Calculating:

$$\begin{aligned} A^2 &= \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 0 & 0 \\ 2 & 1 & 0 \\ 9 & 0 & 1 \end{bmatrix} \end{aligned}$$

Thus

$$A^2 - I = \begin{bmatrix} 8 & 0 & 0 \\ 2 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}.$$

Also

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -4 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

Thus

$$(A^2 - I)(A - 3I) = \begin{bmatrix} 8 & 0 & 0 \\ 2 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & -4 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Exercise 5: Let $\mathbb{C}^{2 \times 2}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$$

and let T be the linear operator on $\mathbb{C}^{2 \times 2}$ defined by $T(A) = BA$. What is the rank of T ? Can you describe T^2 ?

Solution: An (ordered) basis for $\mathbb{C}^{2 \times 2}$ is given by

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

If we identify $\mathbb{C}^{2 \times 2}$ with \mathbb{C}^4 by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a, b, c, d)$$

then since

$$A_{11} \mapsto A_{11} - 4A_{21}$$

$$A_{21} \mapsto -A_{11} + 4A_{21}$$

$$A_{12} \mapsto A_{12} - 4A_{22}$$

$$A_{22} \mapsto -A_{12} + 4A_{22}$$

the matrix of the transformation is given by

$$\begin{bmatrix} 1 & -4 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 4 \end{bmatrix}.$$

To find the rank of T we row-reduce this matrix:

$$\rightarrow \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It has rank two so the rank, so the rank of T is 2.

$T^2(A) = T(T(A)) = T(BA) = B(BA) = B^2A$. Thus T^2 is given by multiplication by a matrix just as T is, but multiplication with B^2 instead of B . Explicitly

$$\begin{aligned} B^2 &= \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -5 \\ -20 & 20 \end{bmatrix}. \end{aligned}$$

Exercise 6: Let T be a linear transformation from \mathbb{R}^3 into \mathbb{R}^2 , and let U be a linear transformation from \mathbb{R}^2 into \mathbb{R}^3 . Prove that the transformation UT is not invertible. Generalize the theorem.

Solution: Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a basis for \mathbb{R}^3 . Then $T(\alpha_1), T(\alpha_2), T(\alpha_3)$ must be linearly dependent in \mathbb{R}^2 . So suppose $b_1T(\alpha_1) + b_2T(\alpha_2) + b_3T(\alpha_3) = 0$ and not all b_1, b_2, b_3 are zero. Then $b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 \neq 0$ and

$$\begin{aligned} & UT(b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3) \\ &= U(T(b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3)) \\ &= U(b_1T(\alpha_1) + b_2T(\alpha_2) + b_3T(\alpha_3)) \\ &= U(0) = 0. \end{aligned}$$

Thus (by the definition at the bottom of page 79) UT is *not* non-singular and thus by Theorem 9, page 81, UT is not invertible.

The obvious generalization is that if $n > m$ and $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $U : \mathbb{R}^m \rightarrow \mathbb{R}^n$ are linear transformations, then UT is not invertible. The proof is an immediate generalization the proof of the special case above, just replace α_3 with \dots, α_n .

Exercise 7: Find two linear operators T and U on \mathbb{R}^2 such that $TU = 0$ but $UT \neq 0$.

Solution: Identify \mathbb{R}^2 with $\mathbb{R}^{2 \times 1}$ and let T and U be given by the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

More precisely, for

$$X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

let T be given by $X \mapsto AX$ and let U be given by $X \mapsto BX$. Thus TU is given by $X \mapsto ABX$ and UT is given by $X \mapsto BAX$. But $BA = 0$ and $AB \neq 0$ so we have the desired example.

Exercise 8: Let V be a vector space over the field F and T a linear operator on V . If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T ? Give an example of a linear operator T on \mathbb{R}^2 such that $T^2 = 0$ but $T \neq 0$.

Solution: If $T^2 = 0$ then the range of T must be contained in the null space of T since if y is in the range of T then $y = Tx$ for some x so $Ty = T(Tx) = T^2x = 0$. Thus y is in the null space of T .

To give an example of an operator where $T^2 = 0$ but $T \neq 0$, let $V = \mathbb{R}^{2 \times 1}$ and let T be given by the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Specifically, for

$$X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

let T be given by $X \mapsto AX$. Since $A \neq 0$, $T \neq 0$. Now T^2 is given by $X \mapsto A^2X$, but $A^2 = 0$. Thus $T^2 = 0$.

Exercise 9: Let T be a linear operator on the finite-dimensional space V . Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when V is not finite-dimensional. (*Hint:* Let $T = D$, be the differentiation operator on the space of polynomial functions.)

Solution: By the comments in the Appendix on functions, at the bottom of page 389, we see that simply because $TU = I$ as functions, then necessarily T is onto and U is one-to-one. It then follows immediately from Theorem 9, page 81, that T is invertible. Now $TT^{-1} = I = TU$ and multiplying on the left by T^{-1} we get $T^{-1}TT^{-1} = T^{-1}TU$ which implies $(I)T^{-1} = (I)U$ and thus $U = T^{-1}$.

Let V be the space of polynomial functions in one variable over \mathbb{R} . Let D be the differentiation operator and let T be the operator “multiplication by x ” (exactly as in Example 11, page 80). As shown in Example 11, $UT = I$ while $TU \neq I$. Thus this example fulfills the requirement.

Exercise 10: Let A be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n \times 1}$ into $F^{m \times 1}$ defined by $TX = AX$. Show that if $m < n$ it may happen that T is onto without being non-singular. Similarly, show that if $m > n$ we may have T non-singular but not onto.

Solution: Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be a basis for $F^{n \times 1}$ and let $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ be a basis for $F^{m \times 1}$. We can define a linear transformation from $F^{n \times 1}$ to $F^{m \times 1}$ uniquely by specifying where each member of \mathcal{B} goes in $F^{m \times 1}$. If $m < n$ then we can define a linear transformation that maps at least one member of \mathcal{B} to each member of \mathcal{B}' and maps at least two members of \mathcal{B} to the same member of \mathcal{B}' . Any linear transformation so defined must necessarily be onto without being one-to-one. Similarly, if $m > n$ then we can map each member of \mathcal{B} to a unique member of \mathcal{B}' with at least one member of \mathcal{B}' not mapped to by any member of \mathcal{B} . Any such transformation so defined will necessarily be one-to-one but not onto.

Exercise 11: Let V be a finite-dimensional vector space and let T be a linear operator on V . Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.

Solution: Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis for V . Then the rank of T is the number of linearly independent vectors in the set $\{T\alpha_1, \dots, T\alpha_n\}$. Suppose the rank of T equals k and suppose WLOG that $\{T\alpha_1, \dots, T\alpha_k\}$ is a linearly independent set (it might be that $k = 1$, pardon the notation). Then $\{T\alpha_1, \dots, T\alpha_k\}$ give a basis for the range of T . It follows that $\{T^2\alpha_1, \dots, T^2\alpha_k\}$ span the range of T^2 and since the dimension of the range of T^2 is also equal to k , $\{T^2\alpha_1, \dots, T^2\alpha_k\}$ must be a basis for the range of T^2 . Now suppose v is in the range of T . Then $v = c_1T\alpha_1 + \dots + c_kT\alpha_k$. Suppose v is also in the null space of T . Then $0 = T(v) = T(c_1T\alpha_1 + \dots + c_kT\alpha_k) = c_1T^2\alpha_1 + \dots + c_kT^2\alpha_k$. But $\{T^2\alpha_1, \dots, T^2\alpha_k\}$ is a basis, so $T^2\alpha_1, \dots, T^2\alpha_k$ are linearly independent, thus it must be that $c_1 = \dots = c_k = 0$, which implies $v = 0$. Thus we have shown that if v is in both the range of T and the null space of T then $v = 0$, as required.

Exercise 12: Let p, m , and n be positive integers and F a field. Let V be the space of $m \times n$ matrices over F and W the space of $p \times n$ matrices over F . Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.

Solution: We showed in Exercise 2.3.12, page 49, that the dimension of V is mn and the dimension of W is pn . By Theorem 9 page (iv) we know that an invertible linear transformation must take a basis to a basis. Thus if there's an invertible linear transformation between V and W it must be that both spaces have the same dimension. Thus if T is invertible then $pn = mn$ which implies $p = m$. The matrix B is then invertible because the assignment $B \mapsto BX$ is one-to-one (Theorem 9 (ii), page 81) and non-invertible matrices have non-trivial solutions to $BX = 0$ (Theorem 13, page 23). Conversely, if $p = m$ and B is invertible, then we can define the inverse transformation T^{-1} by $T^{-1}(A) = B^{-1}A$ and it follows that T is invertible.

Section 3.3: Isomorphism

Exercise 1: Let V be the set of complex numbers and let F be the field of real numbers. With the usual operations, V is a vector space over F . Describe explicitly an isomorphism of this space onto \mathbb{R}^2 .

Solution: The natural isomorphism from V to \mathbb{R}^2 is given by $a + bi \mapsto (a, b)$. Since i acts like a placeholder for addition in \mathbb{C} , $(a + bi) + (c + di) = (a + c) + (b + d)i \mapsto (a + c, b + d) = (a, b) + (c, d)$. And $c(a + bi) = ca + cbi \mapsto (ca, cb) = c(a, b)$. Thus this is a linear transformation. The inverse is clearly $(a, b) \mapsto a + bi$. Thus the two spaces are isomorphic as vector spaces over \mathbb{R} .

Exercise 2: Let V be a vector space over the field of complex numbers, and suppose there is an isomorphism T of V into \mathbb{C}^3 . Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be vectors in V such that

$$T\alpha_1 = (1, 0, i), \quad T\alpha_2 = (-2, 1 + i, 0),$$

$$T\alpha_3 = (-1, 1, 1), \quad T\alpha_4 = (\sqrt{2}, i, 3).$$

- (a) Is α_1 in the subspace spanned by α_2 and α_3 ?
- (b) Let W_1 be the subspace spanned by α_1 and α_2 , and let W_2 be the subspace spanned by α_3 and α_4 . What is the intersection of W_1 and W_2 ?
- (c) Find a basis for the subspace of V spanned by the four vectors α_j .

Solution: (a) Since T is an isomorphism, it suffices to determine whether $T\alpha_1$ is contained in the subspace spanned by $T\alpha_2$ and $T\alpha_3$. In other words we need to determine if there is a solution to

$$\begin{bmatrix} -2 & -1 \\ 1+i & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}.$$

To do this we row-reduce the augmented matrix

$$\begin{aligned} \left[\begin{array}{cc|c} -2 & -1 & 1 \\ 1+i & 1 & 0 \\ 0 & 1 & i \end{array} \right] &\rightarrow \left[\begin{array}{cc|c} 1 & 1/2 & -1/2 \\ 1+i & 1 & 0 \\ 0 & 1 & i \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1/2 & -1/2 \\ 0 & 1 & i \\ 1+i & 1 & 0 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cc|c} 1 & 1/2 & -1/2 \\ 0 & 1 & i \\ 0 & 1/2 & 1+i \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{-1-i}{2} \\ 0 & 1 & i \\ 0 & 0 & 0 \end{array} \right] \end{aligned}$$

The zero row on the left of the dividing line has zero also on the right. This means the system has a solution. Therefore we can conclude that α_1 is in the subspace generated by α_2 and α_3 .

(b) Since $T\alpha_1$ and $T\alpha_2$ are linearly independent, and $T\alpha_3$ and $T\alpha_4$ are linearly independent, $\dim(W_1) = \dim(W_2) = 2$. We row-reduce the matrix whose columns are the $T\alpha_i$:

$$\begin{bmatrix} 1 & -2 & -1 & \sqrt{2} \\ 0 & 1+i & 1 & i \\ i & 0 & 1 & 3 \end{bmatrix}$$

which yields

$$\begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & \frac{1-i}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

from which we deduce that $T\alpha_1, T\alpha_2, T\alpha_3, T\alpha_4$ generate a space of dimension three, thus $\dim(W_1 + W_2) = 3$. It then follows (since $\dim(W_1) = \dim(W_2) = 2$) that $\dim(W_1 \cap W_2) = 1$. We also deduce from the row-reduced matrix that

$$2iT\alpha_1 + (i-1)T\alpha_2 + 2T\alpha_3 = 0$$

which implies $T\alpha_3 = -iT\alpha_1 + \frac{1-i}{2}T\alpha_2$ which implies $T\alpha_3 \in TW_1$. Thus $\alpha_3 \in W_1$. Thus $\alpha_3 \in W_1 \cap W_2$. Since $\dim(W_1 \cap W_2) = 1$ it follows that $W_1 \cap W_2 = \mathbb{C}\alpha_3$.

(c) We have determined in part (b) that the $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ span a space of dimension three, and that α_3 is in the space generated by α_1 and α_2 . Thus $\{\alpha_1, \alpha_2, \alpha_4\}$ give a basis for the subspace spanned by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Exercise 3: Let W be the set of all 2×2 complex Hermitian matrices, that is, the set of 2×2 complex matrices A such that $A_{ij} = \overline{A_{ji}}$ (the bar denoting complex conjugation). As we pointed out in Example 6 of Chapter 2, W is a vector space over the field of *real* numbers, under the usual operations. Verify that

$$(x, y, z, t) \rightarrow \begin{bmatrix} t+x & y+iz \\ y-iz & t-x \end{bmatrix}$$

is an isomorphism of \mathbb{R}^4 onto W .

Solution: The function is linear since the four components are all linear combinations of the components of the domain (x, y, z, t) . Identify $\mathbb{C}^{2 \times 2}$ with \mathbb{C}^4 by $A \mapsto (A_{11}, A_{12}, A_{21}, A_{22})$. Then the matrix of the transformation is given by

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

As usual, the transformation is an isomorphism if the matrix is invertible. We row-reduce to verify the matrix is invertible. We will row-reduce the augmented matrix in order to find the inverse explicitly:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right].$$

This reduces to

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -i/2 & i/2 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & 0 & 0 & 1/2 \end{array} \right].$$

Thus the inverse transformation is

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto \left(\frac{x-w}{2}, \frac{y+z}{2}, \frac{i(z-y)}{2}, \frac{x+w}{2} \right).$$

Chapter 6: Elementary Canonical Forms

Section 6.2: Characteristic Values

Exercise 1: In each of the following cases, let T be the linear operator on \mathbb{R}^2 which is represented by the matrix A in the standard ordered basis for \mathbb{R}^2 , and let U be the linear operator on \mathbb{C}^2 represented by A in the standard ordered basis. Find the characteristic polynomial for T and that for U , find the characteristic values of each operator, and for each such characteristic value c find a basis for the corresponding space of characteristic vectors.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$xI - A = \begin{bmatrix} x-1 & 0 \\ 0 & x \end{bmatrix}$$

The characteristic polynomial equals $|xI - A| = x(x-1)$. So $c_1 = 0$, $c_2 = 1$. A basis for W_1 is $\{(0, 1)\}$, $\alpha_1 = (0, 1)$. A basis for W_2 is $\{(1, 0)\}$, $\alpha_2 = (1, 0)$. This is the same whether the base field is \mathbb{R} or \mathbb{C} since the characteristic polynomial factors completely.

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$
$$|xI - A| = \begin{vmatrix} x-2 & -3 \\ 1 & x-1 \end{vmatrix}$$

$= (x-2)(x-1) + 3 = x^2 - 3x + 5$. This is a parabola opening up with vertex $(3/2, 11/4)$. Thus there are no real roots. Using the quadratic formula $c_1 = \frac{3+\sqrt{11}i}{2}$ and $c_2 = \frac{3-\sqrt{11}i}{2}$. To find the a characteristic vector for c_1 we solve

$$\begin{bmatrix} \frac{-1+\sqrt{11}i}{2} & 3 \\ -1 & \frac{1+\sqrt{11}i}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the characteristic vector $\alpha_1 = (\frac{1+\sqrt{11}i}{2}, 1)$. To find the a characteristic vector for c_2 we solve

$$\begin{bmatrix} \frac{-1-\sqrt{11}i}{2} & 3 \\ -1 & \frac{1-\sqrt{11}i}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the characteristic vector $\alpha_2 = (\frac{1-\sqrt{11}i}{2}, 1)$.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$|xI - A| = \begin{vmatrix} x-1 & 1 \\ 1 & x-1 \end{vmatrix} = (x-1)^2 - 1 = x(x-2)$. So $c_1 = 0$ for which $\alpha_1 = (1, -1)$. And $c_2 = 2$ for which $\alpha_2 = (1, 1)$. This is the same in both \mathbb{R} and \mathbb{C} since the characteristic polynomial factors completely.

Exercise 2: Let F be an n -dimensional vector space over F . What is the characteristic polynomial of the identity operator on V ? What is the characteristic polynomial for the zero operator?

Solution: The identity operator can be represented by the $n \times n$ identity matrix I . The characteristic polynomial of the identity operator is therefore $(x - 1)^n$. The zero operator is represented by the zero matrix in any basis. Thus the characteristic polynomial of the zero operator is x^n .

Exercise 3: Let A be an $n \times n$ triangular matrix over the field F . Prove that the characteristic values of A are the diagonal entries of A , i.e., the scalars A_{ii} .

Solution: The determinant of a triangular matrix is the product of the diagonal entries. Thus $|xI - A| = \prod (x - a_{ii})$.

Exercise 4: Let T be the linear operator of \mathbb{R}^3 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}.$$

Prove that T is diagonalizable by exhibiting a basis for \mathbb{R}^3 , each vector for which is a characteristic vector of T .

Solution:

$$\begin{aligned} |xI - A| &= \begin{vmatrix} x+9 & -4 & -4 \\ 8 & x-3 & -4 \\ 16 & -8 & x-7 \end{vmatrix} \\ &= \begin{vmatrix} x+9 & 0 & -4 \\ 8 & x+1 & -4 \\ 16 & -x-1 & x-7 \end{vmatrix} \\ &= (x+1) \begin{vmatrix} x+9 & 0 & -4 \\ 8 & 1 & -4 \\ 16 & -1 & x-7 \end{vmatrix} \\ &= (x+1) \begin{vmatrix} x+9 & 0 & -4 \\ 8 & 1 & -4 \\ 24 & 0 & x-11 \end{vmatrix} \\ &= (x+1) \begin{vmatrix} x+9 & -4 \\ 24 & x-11 \end{vmatrix} \end{aligned}$$

$= (x+1)[(x+9)(x-11) + 96] = (x+1)(x^2 - 2x - 3) = (x+1)(x-3)(x+1) = (x+1)^2(x-3)$. Thus $c_1 = -1$, $c_2 = 3$. For c_1 , $xI - A$ equals

$$= \begin{bmatrix} 8 & -4 & -4 \\ 8 & -4 & -4 \\ 16 & -8 & -8 \end{bmatrix}$$

This matrix evidently has rank one. Thus the null space has rank two. The two characteristic vectors $(1, 2, 0)$ and $(1, 0, 2)$ are independent, so they form a basis for W_1 . For c_2 , $xI - A$ equals

$$= \begin{bmatrix} -12 & -4 & -4 \\ 8 & 0 & -4 \\ 16 & -8 & -4 \end{bmatrix}$$

This is row equivalent to

$$= \begin{bmatrix} 1 & -0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the null space one dimensional and is given by $(z/2, z/2, z)$. So $(1, 1, 2)$ is a characteristic vector and a basis for W_2 . By Theorem 2 (ii) T is diagonalizable.

Exercise 5: Let

$$\begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}.$$

Is A similar over the field \mathbb{R} to a diagonal matrix? Is A similar over the field \mathbb{C} to a diagonal matrix?

Solution:

$$\begin{aligned} &= \begin{vmatrix} x-6 & 3 & 2 \\ -4 & x+1 & 2 \\ -10 & 5 & x+3 \end{vmatrix} \\ &= \begin{vmatrix} x-6 & 3 & 2 \\ -x+2 & x-2 & 0 \\ -10 & 5 & x+3 \end{vmatrix} \\ &= (x-2) \begin{vmatrix} x-6 & 3 & 2 \\ -1 & 1 & 0 \\ -10 & 5 & x+3 \end{vmatrix} \\ &= (x-2) \begin{vmatrix} x-3 & 3 & 2 \\ 0 & 1 & 0 \\ -5 & 5 & x+3 \end{vmatrix} \end{aligned}$$

$= (x-2)((x-3)(x+3)+10) = (x-2)(x^2+1)$. Since this is not a product of linear factors over \mathbb{R} , by Theorem 2, page 187, A is not diagonalizable over \mathbb{R} . Over \mathbb{C} this factors to $(x-2)(x-i)(x+i)$. Thus over \mathbb{C} the matrix A has three distinct characteristic values. The space of characteristic vectors for a given characteristic value has dimension at least one. Thus the sum of the dimensions of the W_i 's must be at least n . It cannot be greater than n so it must equal n exactly. Thus A is diagonalizable over \mathbb{C} .

Exercise 6: Let T be the linear operator on \mathbb{R}^4 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}.$$

Under what conditions on a , b , and c is T diagonalizable?

Solution:

$$|xI - A| = \begin{vmatrix} x & 0 & 0 & 0 \\ -a & x & 0 & 0 \\ 0 & -b & x & 0 \\ 0 & 0 & -c & x \end{vmatrix}$$

$= x^4$. Therefore there is only one characteristic value $c_1 = 0$. Thus $c_1 I - A = A$ and W_1 is the null space of A . So A is diagonalizable $\Leftrightarrow \dim(W) = 4 \Leftrightarrow A$ is the zero matrix $\Leftrightarrow a = b = c = 0$.

Exercise 7: Let T be the linear operator on the n -dimensional vector space V , and suppose that T has n *distinct* characteristic values. Prove that T is diagonalizable.

Solution: The space of characteristic vectors for a given characteristic value has dimension at least one. Thus the sum of the dimensions of the W_i 's must be at least n . It cannot be greater than n so it must equal n exactly. Thus by Theorem 2, T is diagonalizable.

Exercise 8: Let A and B be $n \times n$ matrices over the field F . Prove that if $(I - AB)$ is invertible, then $I - BA$ is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

Solution:

$$\begin{aligned} & (I - BA)(I + B(I - AB)^{-1}A) \\ &= I - BA + B(I - AB)^{-1}A - BAB(I - AB)^{-1}A \\ &= I - B(A - (I - AB)^{-1}A + AB(I - AB)^{-1}A) \\ &= I - B(I - (I - AB)^{-1} + AB(I - AB)^{-1}A) \\ &= I - B(I - (I - AB)(I - AB)^{-1})A \\ &= I - B(I - I)A \\ &= I. \end{aligned}$$

Exercise 9: Use the result of Exercise 8 to prove that, if A and B are $n \times n$ matrices over the field F , then AB and BA have precisely the same characteristic values in F .

Solution: By Theorem 3, page 154, $\det(AB) = \det(A)\det(B)$. Thus AB is singular $\Leftrightarrow BA$ is singular. Therefore 0 is a characteristic value of $AB \Leftrightarrow 0$ is a characteristic value of BA . Now suppose the characteristic value c of AB is not equal to zero.

$$\text{Then } |cI - AB| = 0 \Leftrightarrow c^n |I - \frac{1}{c}AB| = 0 \stackrel{\text{by \#8}}{\Leftrightarrow} c^n |I - \frac{1}{c}BA| = 0 \Leftrightarrow |cI - BA| = 0.$$

Exercise 10: Suppose that A is a 2×2 matrix with real entries which is symmetric ($A^t = A$). Prove that A is similar over \mathbb{R} to a diagonal matrix.

Solution: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So $|xI - A| = \begin{vmatrix} x-a & -b \\ -b & x-a \end{vmatrix} = (x-a)^2 - b^2 = (x-a-b)(x-a+b)$. So $c_1 = a+b$, $c_2 = a-b$. If $b = 0$ then A is already diagonal. If $b \neq 0$ then $c_1 \neq c_2$ so by Exercise 7 A is diagonalizable.

Exercise 11: Let N be a 2×2 complex matrix such that $N^2 = 0$. Prove that either $N = 0$ or N is similar over \mathbb{C} to

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Solution: Suppose $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now $N^2 = 0 \Rightarrow \begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ are characteristic vectors for the characteristic value 0.

If $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ are linearly independent then W_1 has rank two and N is diagonalizable to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. If $PNP^{-1} = 0$ then

$N = P^{-1}0P = 0$ so in this case N itself is the zero matrix. This contradicts the assumption that $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ are linearly independent.

So we can assume that $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ are linearly dependent. If both equal the zero vector then $N = 0$. So we can assume at least one vector is non-zero. If $\begin{bmatrix} b \\ d \end{bmatrix}$ is the zero vector then $N = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$. So $N^2 = 0 \Rightarrow a^2 = 0 \rightarrow a = 0$. Thus $N = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$. In this case N is similar to $N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ via the matrix $P = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$. Similarly if $\begin{bmatrix} a \\ c \end{bmatrix}$ is the zero vector, then $N^2 = 0$ implies $d^2 = 0$ implies $d = 0$ so $N = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. In this case N is similar to $N = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$ via the matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is similar to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ as above.

By the above we can assume neither $\begin{bmatrix} a \\ c \end{bmatrix}$ or $\begin{bmatrix} b \\ d \end{bmatrix}$ is the zero vector. Since they are linearly dependent we can assume $\begin{bmatrix} b \\ d \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix}$ so $N = \begin{bmatrix} a & ax \\ c & cx \end{bmatrix}$. So $N^2 = 0$ implies

$$a(a + cx) = 0$$

$$c(a + cx) = 0$$

$$ax(a + cx) = 0$$

$$cx(a + cx) = 0.$$

We know that at least one of a or c is not zero. If $a = 0$ then since $c \neq 0$ it must be that $x = 0$. So in this case $N = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ which is similar to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ as before. If $a \neq 0$ then $x \neq 0$ else $a(a + cx) = 0$ implies $a = 0$. Thus $a + cx = 0$ so $N = \begin{bmatrix} a & ax \\ -a/x & -a \end{bmatrix}$. This is similar to $\begin{bmatrix} a & a \\ -a & -a \end{bmatrix}$ via $P = \begin{bmatrix} \sqrt{x} & 0 \\ 0 & 1/\sqrt{x} \end{bmatrix}$. And $\begin{bmatrix} a & a \\ -a & -a \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix}$ via $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$. And this finally is similar to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ as before.

Exercise 12: Use the result of Exercise 11 to prove the following: If A is a 2×2 matrix with complex entries, then A is similar over \mathbb{C} to a matrix of one of the two types

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix}.$$

Solution: Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since the base field is \mathbb{C} the characteristic polynomial $p(x) = (x - c_1)(x - c_2)$. If $c_1 \neq c_2$ then A is diagonalizable by Exercise 7. If $c_1 = c_2$ then $p(x) = (x - c_1)^2$. If W has dimension two then A is diagonalizable by Theorem 2. Thus we will be done if we show that if $p(x) = (x - c_1)^2$ and $\dim(W_1) = 1$ then A is similar to $\begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix}$.

We will need the following three identities:

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & 0 \\ 1 & d \end{bmatrix} \quad \text{via } p = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \quad (20)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a-b & c-d \\ b & d \end{bmatrix} \quad \text{via } p = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad (21)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & xb \\ c/x & d \end{bmatrix} \quad \text{via } p = \begin{bmatrix} \sqrt{x} & 0 \\ 0 & 1/\sqrt{x} \end{bmatrix} \text{ for } x \neq 0. \quad (22)$$

Now we know in this case that A is not diagonalizable. If $d \neq 0$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & bc/d \\ d & d \end{bmatrix}$ by (22) with $x = c/d$ and this in turn is similar to $\begin{bmatrix} a - bc/d & 0 \\ -a + 2bc/d & d \end{bmatrix}$ by (21).

Now we know the diagonal entries are the characteristic values, which are equal. Thus $a - \frac{bc}{d} = d$. So this equals $\begin{bmatrix} d & 0 \\ x & d \end{bmatrix}$ where $x = -a + \frac{2bc}{d}$ and we know $x \neq 0$ since A is not diagonalizable. Thus $A \sim \begin{bmatrix} d & 0 \\ 1 & d \end{bmatrix}$ by (20). Now suppose $d = 0$. Then $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & c \\ b & a \end{bmatrix}$ via $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. If $b = 0$ then $A = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$ and again since A has equal characteristic values it must be that $a = 0$. So $A = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ which is similar to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ via $P = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$. So assume $b \neq 0$. Then $A \sim \begin{bmatrix} 0 & c \\ b & a \end{bmatrix}$ and we can argue exact as above were $d \neq 0$.

Exercise 13: Let V be the vector space of all functions from \mathbb{R} into \mathbb{R} which are continuous, i.e., the space of continuous real-valued functions on the real line. Let T be the linear operator on V defined by

$$(Tf)(x) = \int_0^x f(t)dt.$$

Prove that T has no characteristic values.

Solution: Suppose $\exists c$ such that $Tf = cf \forall f$. Then $\int_0^x f(t)dt = cf(x)$. Let $f(x) = 1$. Then we must have $\int_0^x 1dt = c \cdot 1$, which implies $x = c$. In other words this implies the functions $f(x) = x$ and $g(x) = c$ are the same, which they are not because one is constant and the other is not. This therefore is a contradiction. Thus it is impossible that $\exists c$ such that $Tf = cf \forall f$. Thus T has no characteristic values.

Exercise 14: Let A be an $n \times n$ diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1} \cdots (x - c_k)^{d_k},$$

where c_1, \dots, c_k are distinct. Let V be the space of $n \times n$ matrices B such that $AB = BA$. Prove that the dimension of V is $d_1^2 + \cdots + d_k^2$.

Solution: Write

$$A = \begin{bmatrix} c_1 I & & & \\ & c_2 I & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & c_k I \end{bmatrix}.$$

Write

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \cdots & B_{kk} \end{bmatrix}$$

where B_{ij} has dimension $d_i \times d_j$. Then $AB = BA$ implies

$$\begin{bmatrix} c_1 B_{11} & c_1 B_{12} & \cdots & c_1 B_{1k} \\ c_2 B_{21} & c_2 B_{22} & \cdots & c_2 B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_k B_{k1} & c_k B_{k2} & \cdots & c_k B_{kk} \end{bmatrix} = \begin{bmatrix} c_1 B_{11} & c_2 B_{12} & \cdots & c_k B_{1k} \\ c_1 B_{21} & c_2 B_{22} & \cdots & c_k B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 B_{k1} & c_2 B_{k2} & \cdots & c_k B_{kk} \end{bmatrix}$$

Thus $c_i \neq c_j$ for $i \neq j$ implies $B_{ij} = 0$ for $i \neq j$, while $B_{11}, B_{22}, \dots, B_{kk}$ can be arbitrary. The dimension of B_{ii} is therefore d_i^2 thus the dimension of the space of all such B_{ii} 's is $d_1^2 + d_2^2 + \dots + d_k^2$.

Exercise 15: Let V be the space of $n \times n$ matrices over F . Let A be a fixed $n \times n$ matrix over F . Let T be the linear operator 'left multiplication by A ' on V . Is it true that A and T have the same characteristic values?

Solution: Yes. Represent an element of V as a column vector by stacking the columns of V on top of each other, with the

first column on top. Then the matrix for T is given by
$$\begin{bmatrix} A & & & \\ & A & 0 & \\ & 0 & \ddots & \\ & & & A \end{bmatrix}$$
. By the argument on page 157 the determinant of

this matrix is $\det(A)^n$. Thus if p is the characteristic polynomial of A then p^n is the characteristic polynomial of T . Thus they have exactly the same roots and thus they have exactly the same characteristic values.

Section 6.3: Annihilating Polynomials

Page 198: Typo in Exercise 11, "Section 6.1" should be "Section 6.2".

Exercise 1: Let V be a finite-dimensional vector space. What is the minimal polynomial for the identity operator on V ? What is the minimal polynomial for the zero operator?

Solution: The minimal polynomial for the identity operator is $x - 1$. It annihilates the identity operator and the monic zero degree polynomial $p(x) = 1$ does not, so it must be the minimal polynomial. The minimal polynomial for the zero operator is x . It is a monic polynomial that annihilates the zero operator and again the monic zero degree polynomial $p(x) = 1$ does not, so it must be the minimal polynomial.

Exercise 2: Let a, b and c be elements of a field F , and let A be the following 3×3 matrix over F :

$$A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}.$$

Prove that the characteristic polynomial for A is $x^3 - ax^2 - bx - c$ and that this is also the minimal polynomial for A .

Solution: The characteristic polynomial is

$$\begin{vmatrix} x & 0 & -c \\ -1 & x & -b \\ 0 & -1 & x-a \end{vmatrix} = \begin{vmatrix} x & 0 & -c \\ -1 & 0 & x^2 - ax - b \\ 0 & -1 & x-a \end{vmatrix} = 1 \cdot \begin{vmatrix} x & -c \\ -1 & x^2 - ax - b \end{vmatrix} = x^3 - ax^2 - bx - c.$$

Now for any $r, s \in F$

$$\begin{aligned} A^2 + rA + sI &= \begin{bmatrix} 0 & c & ac \\ 0 & b & c+ba \\ 1 & a & b+a^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & rc \\ r & 0 & rb \\ 0 & r & ra \end{bmatrix} + \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \\ &= \begin{bmatrix} s & c & ac+rc \\ r & b+s & c+ba+br \\ 1 & a+r & b+a^2+ra+s \end{bmatrix} \neq 0. \end{aligned}$$

Thus $f(A) \neq 0$ for all $f \in F[x]$ such that $\deg(f) = 2$. Thus the minimum polynomial cannot have degree two, it must therefore have degree three. Since it divides $x^3 - ax^2 - bx - c$ it must equal $x^3 - ax^2 - bx - c$.

Exercise 3: Let A be the 4×4 real matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

Show that the characteristic polynomial for A is $x^2(x-1)^2$ and that it is also the minimal polynomial.

Solution: The characteristic polynomial equals

$$\begin{vmatrix} x-1 & -1 & 0 & 0 \\ 1 & x+1 & 0 & 0 \\ 2 & 2 & x-2 & -1 \\ -1 & -1 & 1 & x \end{vmatrix} = \begin{vmatrix} x-1 & -1 \\ 1 & x+1 \end{vmatrix} \cdot \begin{vmatrix} x-2 & -1 \\ 1 & x \end{vmatrix} \quad \text{by (5-20) page 158}$$

$$= x^2(x^2 - 2x + 1) = x^2(x-1)^2.$$

The minimum polynomial is clearly not linear, thus the minimal polynomial is one of $x^2(x-1)^2$, $x^2(x-1)$, $x(x-1)^2$ or $x(x-1)$. We will plug A in to the first three and show it is not zero. It will follow that the minimum polynomial must be $x^2(x-1)^2$.

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & -1 \end{bmatrix}$$

$$A - I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

and

$$(A - I)^2 = \begin{bmatrix} -1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus

$$A^2(A - I) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \neq 0$$

$$A(A - I)^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0$$

and

$$A(A - I) = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -2 & -2 \end{bmatrix} \neq 0.$$

Thus the minimal polynomial must be $x^2(x-1)^2$.

Exercise 4: Is the matrix A of Exercise 3 similar over the field of complex numbers to a diagonal matrix?

Solution: Not diagonalizable, because for characteristic value $c = 0$ the matrix $A - cI = A$ and A is row equivalent to

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has rank three. So the null space has dimension one. So if W is the null space for $A - cI$ then W has dimension one, which is less than the power of x in the characteristic polynomial. So by Theorem 2, page 187, A is not diagonalizable.

Exercise 5: Let V be an n -dimensional vector space and let T be a linear operator on V . Suppose that there exists some positive integer k so that $T^k = 0$. Prove that $T^n = 0$.

Solution: $T^k = 0 \Rightarrow$ the only characteristic value is zero. We know the minimal polynomial divides this so the minimal polynomial is of the form t^r for some $1 \leq r \leq n$. Thus by Theorem 3, page 193, the characteristic polynomial's only root is zero, and the characteristic polynomial has degree n . So the characteristic polynomial equals t^n . By Theorem 4 (Caley-Hamilton) $T^n = 0$.

Exercise 6: Find a 3×3 matrix for which the minimal polynomial is x^2 .

Solution: If $A^2 = 0$ and $A \neq 0$ then the minimal polynomial is x or x^2 . So any $A \neq 0$ such that $A^2 = 0$ has minimal polynomial x^2 . E.g.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Exercise 7: Let n be a positive integer, and let V be the space of polynomials over \mathbb{R} which have degree at most n (throw in the 0-polynomial). Let D be the differentiation operator on V . What is the minimal polynomial for D ?

Solution: $1, x, x^2, \dots, x^n$ is a basis.

$$\begin{aligned} 1 &\mapsto 0 \\ x &\mapsto 1 \\ x^2 &\mapsto 2x \\ &\vdots \\ x^n &\mapsto nx^{n-1} \end{aligned}$$

The matrix for D is therefore

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

Suppose A is a matrix such that $a_{ij} = 0$ except when $j = i + 1$. Then A^2 has $a_{ij} = 0$ except when $j = i + 2$. A^3 has $a_{ij} = 0$ except when $j = i + 3$. Etc., where finally $A^n = 0$. Thus if $a_{ij} \neq 0 \forall j = i + 1$ then $A^k \neq 0$ for $k < n$ and $A^n = 0$. Thus the minimum polynomial divides x^n and cannot be x^k for $k < n$. Thus the minimum polynomial is x^n .

Exercise 8: Let P be the operator on \mathbb{R}^2 which projects each vector onto the x -axis, parallel to the y -axis: $P(x, y) = (x, 0)$. Show that P is linear. What is the minimal polynomial for P ?

Solution: P can be given in the standard basis by left multiplication by $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since P is given by left multiplication by a matrix, P is clearly linear. Since A is diagonal, the characteristic values are the diagonal values. Thus the characteristic values of A are 0 and 1. The characteristic polynomial is a degree two monic polynomial for which both 0 and 1 are roots. Therefore the characteristic polynomial is $x(x-1)$. If the characteristic polynomial is a product of distinct linear terms then it must equal the minimal polynomial. Thus the minimal polynomial is also $x(x-1)$.

Exercise 9: Let A be an $n \times n$ matrix with characteristic polynomial

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}.$$

Show that

$$c_1 d_1 + \cdots + c_k d_k = \text{trace}(A).$$

Solution: Suppose A is $n \times n$. Claim: $|xI - A| = x^n + \text{trace}(A)x^{n-1} + \cdots$. Proof by induction: case $n = 2$. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $|xI - A| = x^2 + (a+d)x + (ad - bc)$. The trace of A is $a+d$ so we have established the claim for the case $n = 2$. Suppose true for up to $n-1$. Let $r = a_{22} + a_{33} + \cdots + a_{nn}$. Then

$$\begin{vmatrix} x - a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & x - a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & x - a_{nn} \end{vmatrix}$$

Now expanding by minors using the first column, and using induction, we get that this equals

$$\begin{aligned} & (x - a_{11})(x^{n-1} - r x^{n-2} + \cdots) \\ & - a_{21}(\text{polynomial of degree } n-2) \\ & + a_{31}(\text{polynomial of degree } n-2) \\ & + \cdots \\ & = x^n + (r + a_{11})x^{n-1} + \text{polynomial of degree at most } n-2 \\ & = x^n - \text{tr}(A)x^{n-1} + \cdots \end{aligned}$$

Now if $f(x) = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$ then the coefficient of x^{n-1} is $c_1 d_1 + \cdots + c_k d_k$ so it must be that $c_1 d_1 + \cdots + c_k d_k = \text{tr}(A)$.

Exercise 10: Let V be the vector space of $n \times n$ matrices over the field F . Let A be a fixed $n \times n$ matrix. Let T be the linear operator on V defined by

$$T(B) = AB.$$

Show that the minimal polynomial for T is the minimal polynomial for A .

Solution: If we represent a $n \times n$ matrix as a column vector by stacking the columns of the matrix on top of each other, with the first column on the top, then the transformation T is represented in the standard basis by the matrix

$$M = \begin{bmatrix} A & & & \\ & A & 0 & \\ & 0 & \ddots & \\ & & & A \end{bmatrix}.$$

And since

$$f(M) = \begin{bmatrix} f(A) & & & \\ & f(A) & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & f(A) \end{bmatrix}$$

it is evident that $f(M) = 0 \Leftrightarrow f(A) = 0$.

Exercise 11: Let A and B be $n \times n$ matrices over the field F . According to Exercise 9 of Section 6.2, the matrices AB and BA have the same characteristic values. Do they have the same characteristic polynomial? Do they have the same minimal polynomial?

Solution: In Exercise 9 Section 6.2 we showed $|xI - AB| = 0 \Leftrightarrow |xI - BA| = 0$. Thus we have two monic polynomials of degree n with exactly the same roots. Thus they are equal. So the characteristic polynomials are equal. But the minimum polynomials need not be equal. To see this let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so the minimal polynomial of BA is x and the minimal polynomial of AB is clearly not x (it is in fact x^2).

Section 6.4: Invariant Subspaces

Exercise 1: Let T be the linear operator on \mathbb{R}^2 , the matrix of which in the standard ordered basis is

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$

(a) Prove that the only subspaces of \mathbb{R}^2 invariant under T are \mathbb{R}^2 and the zero subspace.

(b) If U is the linear operator on \mathbb{C}^2 , the matrix of which in the standard ordered basis is A , show that U has 1-dimensional invariant subspaces.

Solution: (a) The characteristic polynomial equals $\begin{vmatrix} x-1 & 1 \\ -2 & x-2 \end{vmatrix} = (x-1)(x-2)+2 = x^2-3x+4$. This is a parabola opening upwards with vertex $(3/2, 7/4)$, so it has no real roots. If T had an invariant subspace it would have to be 1-dimensional and T would therefore have a characteristic value.

(b) Over \mathbb{C} the characteristic polynomial factors into two linears. Therefore over \mathbb{C} , T has two characteristic values and therefore has at least one characteristic vector. The subspace generated by a characteristic vector is a 1-dimensional subspace.

Exercise 2: Let W be an invariant subspace for T . Prove that the minimal polynomial for the restriction operator T_W divides the minimal polynomial for T , without referring to matrices.

Solution: The minimum polynomial of T_W divides any polynomial $f(t)$ where $f(T_W) = 0$. If f is the minimum polynomial for T then $F(T)v = 0 \forall v \in V$. Therefore, $f(T)w = 0 \forall w \in W$. So $f(T_W)w = 0 \forall w \in W$ since by definition $f(T_W)w = f(T)w$ for $w \in W$. Therefore, $f(T_W) = 0$. Therefore the minimum polynomial for T_W divides f .

Exercise 3: Let c be a characteristic value of T and let W be the space of characteristic vectors associated with the characteristic value c . What is the restriction operator T_W ?

Solution: For $w \in W$ the transformation $T(w) = cw$. Thus T_W is diagonalizable with single characteristic value c . In other

words under which it is represented by the matrix

$$\begin{bmatrix} c & & & \\ & c & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & c \end{bmatrix}$$

where there are $\dim(W)$ c 's on the diagonal.

Exercise 4: Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{bmatrix}.$$

Is A similar over the field of real numbers to a triangular matrix? If so, find such a triangular matrix.

Solution:

$$A^2 = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ -2 & 2 & -2 \end{bmatrix}.$$

And $A^3 = 0$. Thus the minimal polynomial x^3 and the only characteristic value is 0. We now follow the constructive proof

of Theorem 5. $W = \{0\}$, α_1 a characteristic vector of A is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. We need α_2 such that $A\alpha_2 \in \{\alpha_1\}$. $\alpha_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ satisfies

$A\alpha_2 = \alpha_1$. Now need α_3 such that $A\alpha_3 \in \{\alpha_1, \alpha_2\}$. $\alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ satisfies $A\alpha_3 = 2\alpha_1 + 2\alpha_2$. Thus with respect to the basis

$\{\alpha_1, \alpha_2, \alpha_3\}$ the transformation corresponding to A is $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Exercise 5: Every matrix A such that $A^2 = A$ is similar to a diagonal matrix.

Solution: $A^2 = A \Rightarrow A$ satisfies the polynomial $x^2 - x = x(x-1)$. Therefore the minimum polynomial of A is either x , $x-1$ or $x(x-1)$. In all three cases the minimum polynomial factors into distinct linears. Therefore, by Theorem 6 A is diagonalizable.

Exercise 6: Let T be a diagonalizable linear operator on the n -dimensional vector space V , and let W be a subspace which is invariant under T . Prove that the restriction operator T_W is diagonalizable.

Solution: By the lemma on page 80 the minimum polynomial for T_W divides the minimum polynomial for T . Now T diagonalizable implies (by Theorem 6) that the minimum polynomial for T factors into distinct linears. Since the minimum polynomial for T_W divides it, it must also factor into distinct linears. Thus by Theorem 6 again T_W is diagonalizable.

Exercise 7: Let T be a linear operator on a finite-dimensional vector space over the field of complex numbers. Prove that T is diagonalizable if and only if T is annihilated by some polynomial over \mathbb{C} which has distinct roots.

Solution: If T is diagonalizable then its minimum polynomial is a product of distinct linear factors, and the minimal polynomial annihilates T . This proves " \Rightarrow ". Now suppose T is annihilated by a polynomial over \mathbb{C} with distinct roots. Since the base field is \mathbb{C} this polynomial factors completely into distinct linear factors. Since the minimum polynomial divides this polynomial the minimum polynomial factors completely into distinct linear factors. Thus by Theorem 6, T is diagonalizable.

Exercise 8: Let T be a linear operator on V . If every subspace of V is invariant under T , then T is a scalar multiple of the identity operator.

Solution: Let $\{\alpha_i\}$ be a basis. The subspace generated by α_i is invariant thus $T\alpha_i$ is a multiple of α_i . Thus α_i is a characteristic vector since $T\alpha_i = c_i\alpha_i$ for some c_i . Suppose $\exists i, j$ such that $c_i \neq c_j$. Then $T(\alpha_i + \alpha_j) = T\alpha_i + T\alpha_j = c_i\alpha_i + c_j\alpha_j = c(\alpha_i + \alpha_j)$. Since the subspace generated by $\{\alpha_i, \alpha_j\}$ is invariant under T . Thus $c_i = c$ and $c_j = c$ since coefficients of linear combinations of basis vectors are unique. Thus $T\alpha_i = c\alpha_i \forall i$. Thus T is c times the identity operator.

Exercise 9: Let T be the indefinite inntegral operator

$$(Tf)(x) = \int_0^x f(t)dt$$

on the space of continuous functions on the interval $[0, 1]$. Is the space of polynomial functions invariant under T ? Ths space of differentiable functions? The space of functions which vanish at $x = 1/2$?

Solution: The integral from 0 to x of a polynomial is again a polynomial, so the space of polynomial functions is invariant under T . The integral from 0 to x of a differntiable function is differentiable, so the space of differentiable functions is invariant under T . Now let $f(x) = x - 1/2$. Then f vanishes at $1/2$ but $\int_0^x f(t)dt = \frac{1}{2}x^2 - \frac{1}{2}x$ which does not vanish at $x = 1/2$. So the space of functions which vanish at $x = 1/2$ is not invariant under T .

Exercise 10: Let A be a 3×3 matrix with real entries. Prove that, if A is not similar over \mathbb{R} to a triangular matrix, then A is similar over \mathbb{C} to a diagonal matrix.

Solution: If A is not similar to a tirangular matrix then the minimum polynomial of A must be of the form $(x - c)(x^2 + ax + b)$ where $x^2 + ax + b$ has no real roots. The roots of $x^2 + ax + b$ are then two non-real complex conjugates z and \bar{z} . Thus over \mathbb{C} the minimum polynomial factors as $(x - c)(x - z)(x - \bar{z})$. Since c is real, c , z and \bar{z} constintute three distinct numbers. Thus by Theorem 6 A is diagonalizable over \mathbb{C} .

Exercise 11: True or false? If the triangular matrix A is similar to a diagonal matrix, then A is already diagonal.

Solution: False. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then A is triangular and not diagonal. The characteristic polynomial is $x(x - 1)$ which has distinct roots, so the minimum polynomial is $x(x - 1)$. Thus by Theorem 6, A is diagonalizable.

Exercise 12: Let T be a linear operator on a finite-dimensional vector space over an algebraically closed field F . Let f be a polynomial over F . Prove that c is a characteristic value of $f(T)$ if and only if $c = f(t)$, where t is a characteristic value of T .

Solution: Since F is algebraically closed, the corollary at the bottom of page 203 implies there's a basis under which T is represented by a triangular matrix A . $A = [a_{ij}]$ where $a_{ij} = 0$ if $i > j$ and the a_{ii} , $i = 1, \dots, n$ are the characteristic values of T . Now $f(A) = [b_{ij}]$ where $b_{ij} = 0$ if $i > j$ and $b_{ii} = f(a_{ii})$ for all $i = 1, \dots, n$. Thus the characteristic values of $f(A)$ are exactly the $f(c)$'s where c is a characteristic value of A . Since $f(A)$ is a matrix representative of $f(T)$ in the same basis, we conclude the same thing about the tranformation T .

Exercise 13: Let V be the space of $n \times n$ matrices over F . Let A be a fixed $n \times n$ matrix over F . Let T and U be the linear operators on V defined by

$$\begin{aligned} T(B) &= AB \\ U(B) &= AB - BA \end{aligned}$$

- (a) True or false? If A is a diagonalizable (over F), then T is diagonalizable.
 (b) True or false? If A is diagonalizable, then U is diagonalizable.

Solution: (a) True by Exercise 10 Section 6.3 page 198 since by Theorem 6 diagonalizability depends entirely on the minimum polynomial.

(b) True. Find a basis so that A is diagonal

$$A = \begin{bmatrix} c_1 & & & \\ & c_2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & c_n \end{bmatrix}.$$

Let $B = [b_{ij}]$. Then $U(B) = AB - BA$. The n^2 matrices B_{ij} such that $b_{ij} \neq 0$ and all other entries equal zero form a basis for V . For any B_{ij} , $U(B_{ij}) = AB_{ij} - B_{ij}A = [d_{i'j'}]$ where $d_{i'j'} = c_{i'}b_{i'j'} - c_{j'}b_{i'j'} = (c_{i'} - c_{j'})b_{i'j'}$. Thus $d_{i'j'} \neq 0$ only when $i' = i$ and $j' = j$. Thus $U(B_{ij}) = (c_i - c_j)B_{ij}$. So $c_i - c_j$ is a characteristic value and B_{ij} is a characteristic vector for all i, j . Thus V has a basis of characteristic vectors for U . Thus U is diagonalizable.

Section 6.5: Simultaneous Triangulation; Simultaneous Diagonalization

Exercise 1: Find an invertible real matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal, where A and B are the real matrices

$$(a) \quad A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -8 \\ 0 & -1 \end{bmatrix}$$

$$(b) \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}.$$

Solution: The proof of Theorem 8 shows that if a 2×2 matrix has two characteristic values then the P that diagonalizes A will necessarily also diagonalize any B that commutes with A .

(a) Characteristic polynomial equals $(x - 1)(x - 2)$. So $c_1 = 1, c_2 = 2$.

$$c_1 : \begin{bmatrix} 0 & -2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_2 : \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{So } P = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P^{-1}BP = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) Characteristic polynomial equals $x(x - 2)$. So $c_1 = 0, c_2 = 2$.

$$c_1 : \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_2 : \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{So } P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P^{-1}BP = \begin{bmatrix} 1-a & 0 \\ 0 & 1+a \end{bmatrix}$$

Exercise 2: Let \mathcal{F} be a commuting family of 3×3 complex matrices. How many linearly independent matrices can \mathcal{F} contain? What about the $n \times n$ case?

Solution: This turns out to be quite a hard question, so I'm not sure what Hoffman & Kunze had in mind. But there's a general theorem from 1905 by I. Schur which says the answer is $\lfloor \frac{n^2}{4} \rfloor + 1$. A simpler proof was published in 1998 by M. Mirzakhani in the American Mathematical Monthly. I have extracted the proof for the case $n = 3$ (which required also extracting the proof for the case $n = 2$).

First we show:

The maximum size of a set of linearly independent commuting triangulizable 2×2 matrices is two (*)

Suppose that $\{A_1, A_2, A_3\}$ are three linearly independent commuting upper-triangular 2×2 matrices. Let V be the space generated by $\{A_1, A_2, A_3\}$. So $\dim(V) = 3$.

Write $A_i = \begin{bmatrix} N_i \\ 0 & M_i \end{bmatrix}$ where N_i is 1×2 . Since $\dim(V) = 3$ it cannot be that all three M_i 's are zero. Assume WLOG that $M_1 \neq 0$. Then $M_2 = c_2 M_1$ and $M_3 = c_3 M_1$ for some constants c_1, c_2 . Let $B_2 = A_2 - c_2 A_1$ and $B_3 = A_3 - c_3 A_1$. Then $\{B_2, B_3\}$ are linearly independent in V .

Write $B_2 = \begin{bmatrix} t_2 \\ 0 & 0 \end{bmatrix}$ and $B_3 = \begin{bmatrix} t_3 \\ 0 & 0 \end{bmatrix}$ where $\{t_2, t_3\}$ are linearly independent 1×2 matrices.

Similarly $\exists B'_2$ and B'_3 in V such that $B'_2 = \begin{bmatrix} 0 & t'_2 \\ 0 & 0 \end{bmatrix}$, $B'_3 = \begin{bmatrix} 0 & t'_3 \\ 0 & 0 \end{bmatrix}$, where $\{t'_2, t'_3\}$ are linearly independent.

Since B_2, B_3, B'_2, B'_3 are all in V , they all commute with each other. Thus $t_i t'_j = 0 \forall i, j$.

Let A be the 2×2 matrix $\begin{bmatrix} t_3 \\ t_2 \end{bmatrix}$. Then $\text{rank}(A) = 2$ but $A t'_2 = 0$ and $A t'_3 = 0$ thus $\text{null}(A) = 2$. Therefore $\text{rank}(A) + \text{null}(A) = 4$. But $\text{rank}(A) + \text{null}(A)$ cannot be greater than $\dim(V) = 2$. This contradiction implies we cannot have $\{A_1, A_2, A_3\}$ all three be commuting linearly independent upper-triangular 2×2 matrices. But we know we can have two commuting linearly independent upper-triangular 2×2 matrices because $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are such a pair.

We now turn to the case $n = 3$. Suppose \mathcal{F} is a commuting family of linearly independent 3×3 matrices with $|\mathcal{F}| = 4$. We know $\exists P$ such that $P^{-1}\mathcal{F}P$ is a family of upper triangular commuting matrices. Let V be the space generated by \mathcal{F} . Then $\dim(V) = 4$. Let A_1, A_2, A_3, A_4 be a linearly independent subset of V . For each $i \exists$ a 2×2 matrix M_i and a 1×3 matrix N_i such that

$$A_i = \begin{bmatrix} N_i \\ 0 & M_i \end{bmatrix}$$

Since the A_i 's commute, for $1 \leq i, j \leq 4$ we have $M_i M_j = M_j M_i$. Suppose W is the vector space spanned by the set $\{M_1, M_2, M_3, M_4\}$ and let $k = \dim(W)$. We know by (*) that $k \leq 2$. Since $\{A_1, A_2, A_3, A_4\}$ are independent we also know $k \geq 1$.

First assume $k = 1$. Then WLOG assume M_1 generates W . Then for $i = 2, 3, 4 \exists n_i$ such that $M_i = n_i M_1$. For $i = 2, 3, 4$ define $B_i = A_i - n_i A_1$. Since $\{A_1, A_2, A_3, A_4\}$ are linearly independent, $\{B_2, B_3, B_4\}$ are linearly independent and $B_i = \begin{bmatrix} t_i \\ 0 \end{bmatrix}$ where t_i is 1×3 and $\{t_2, t_3, t_4\}$ are linearly independent.

Now assume $k = 2$. Then WLOG assume M_1, M_2 generate W . Then for each $i = 3, 4 \exists n_{i1}, n_{i2}$ such that $M_i = n_{i1} M_1 + n_{i2} M_2$. For $i = 3, 4$ define $B_i = A_i - n_{i1} A_1 - n_{i2} A_2$. Then $\{A_1, A_2, A_3, A_4\}$ linearly independent implies $\{B_3, B_4\}$ are linearly independent and for $i = 3, 4, B_i = \begin{bmatrix} t_i \\ 0 \end{bmatrix}$ where t_i is $1 \times n$. Since $\{B_3, B_4\}$ are linearly independent, $\{t_3, t_4\}$ are linearly independent.

Thus in both cases ($k = 1, 2$) we have produced a set of $4 - k$ linearly independent $1 \times n$ matrices $\{t_i\}$ such that $B_i = \begin{bmatrix} t_i \\ 0 \end{bmatrix}$.

By a similar argument we obtain a set of two or three linearly independent $n \times 1$ matrices $\{t'_3, t'_4\}$ or $\{t'_2, t'_3, t'_4\}$ such that $B'_i = [0 \mid t'_i]$ is a matrix in V .

Now since all B_i 's and B'_i 's all belong to the commuting family V , one sees that $t_i t'_j = 0 \forall i, j$.

Let A be the $m \times 4$ matrix ($m = 2$ or 3) such that its i th row is t_i . Since the t_i 's are independent we have $\text{rank}(A) \geq m \geq 2$. On the other hand $A t'_j = 0$ for all j and the t'_j 's are linearly independent. Thus the null space of A has rank greater or equal to the number of t'_j 's. Thus $\text{rank}(A) \geq 2$ and $\text{null}(A) \geq 2$. But since A is 3×3 we know that $\text{rank}(A) + \text{null}(A) = 3$. This contradiction implies the set $\{A_1, A_2, A_3, A_4\}$ cannot be linearly independent.

Now we can achieve three independent such matrices because $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are such a triple.

Exercise 3: Let T be a linear operator on an n -dimensional space, and suppose that T has n distinct characteristic values. Prove that any linear operator which commutes with T is a polynomial in T .

Solution: Since T has n distinct characteristic values, T is diagonalizable (exercise 6.2.7, page 190). Choose a basis \mathcal{B} for which T is represented by a diagonal matrix A . Suppose the linear transformation S commutes with T . Let B be the matrix of S in the basis \mathcal{B} . Then the ij -th entry of AB is $a_{ii}b_{ij}$ and the ij -th entry of BA is $a_{jj}b_{ij}$. Therefore if $a_{ii}b_{ij} = a_{jj}b_{ij}$ and $a_{ii} \neq a_{jj}$, then it must be that $b_{ij} = 0$. So we have shown that B must also be diagonal. So we have to show there exists a polynomial such that $f(a_{ii}) = b_{ii}$ for all $i = 1, \dots, n$. By Section 4.3 there exists a polynomial with this property.

Exercise 4: Let A, B, C , and D be $n \times n$ complex matrices which commute. Let E be the $2n \times 2n$ matrix

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Prove that $\det E = \det(AD - BC)$.

Solution: By the corollary on page 203 we know A, B, C , and D are all triangulable. By Theorem 7 page 207 we know they are simultaneously triangulable. Let P be the matrix that simultaneously triangulates them. Let

$$M = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}.$$

Then

$$M^{-1} = \begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix}.$$

And

$$M^{-1}EM = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix},$$

where A', B', C' , and D' are upper triangular. Now $\det(E) = \det(M^{-1}EM)$. Suppose the result were true for upper triangular matrices A, B, C , and D . Then $\det(E) = \det(M^{-1}EM) = \det(P^{-1}AP \cdot P^{-1}DP - P^{-1}BP \cdot P^{-1}CP) = \det(P^{-1}ADP - P^{-1}BCP) = \det(P^{-1}(AD - BC)P) = \det(AD - BC)$.

Thus it suffices to prove the result for upper triangular matrices. So in what follows we drop the primes and assume A, B, C , and D are upper triangular. We proceed by induction. Suppose first that $n = 1$. Then the theorem is clearly true. Suppose it is true up to $n - 1$.

If A , B , C , and D are upper triangular then it is clear that $\det(AD - BC) = \prod_{i=1}^n (a_{ii}d_{ii} - b_{ii}c_{ii})$. So by induction we assume $\det(E) = \prod_{i=1}^m (a_{ii}d_{ii} - b_{ii}c_{ii})$ whenever E has dimension $2m$ for $m < n$ (of course always assuming A , B , C , and D commute).

Now we can write E in the following form for some A' , B' , C' and D' :

$$E = \left[\begin{array}{cc|cc} a_{11} & A' & b_{11} & B' \\ & \ddots & & \ddots \\ 0 & a_{nn} & 0 & b_{nn} \\ \hline c_{11} & C' & d_{11} & D' \\ & \ddots & & \ddots \\ 0 & c_{nn} & 0 & d_{nn} \end{array} \right]$$

Expanding E by cofactors of the 1st column gives

$$\det(E) = a_{11} \left| \begin{array}{cc|cc} a_{22} & A'' & 0 & b_{22} & B'' \\ & \ddots & & \ddots & \\ 0 & a_{nn} & 0 & b_{nn} & \\ \hline c_{22} & C'' & d_{11} & D'' \\ & \ddots & & \ddots & \\ 0 & c_{nn} & 0 & d_{nn} & \end{array} \right| + (-1)^n c_{11} \left| \begin{array}{cc|cc} a_{22} & A' & b_{11} & B' \\ & \ddots & & \ddots & \\ 0 & a_{nn} & 0 & b_{nn} & \\ \hline c_{22} & C'' & 0 & d_{22} & D'' \\ & \ddots & & \ddots & \\ 0 & c_{nn} & 0 & d_{nn} & \end{array} \right|$$

for some A'' , B'' , C'' and D'' .

The $n + 1$ column of each of these matrices has only one non-zero element. So we next expand by cofactors of the $n + 1$ -th column of each matrix, which gives

$$\begin{aligned} & (-1)^{2n} a_{11} d_{11} \left| \begin{array}{cc|cc} a_{22} & A'' & b_{22} & B'' \\ & \ddots & & \ddots & \\ 0 & a_{nn} & 0 & b_{nn} & \\ \hline c_{22} & C'' & d_{22} & D'' \\ & \ddots & & \ddots & \\ 0 & c_{nn} & 0 & d_{nn} & \end{array} \right| + (-1)^{2n+1} c_{11} b_{11} \left| \begin{array}{cc|cc} a_{22} & A'' & b_{22} & B'' \\ & \ddots & & \ddots & \\ 0 & a_{nn} & 0 & b_{nn} & \\ \hline c_{22} & C'' & d_{22} & D'' \\ & \ddots & & \ddots & \\ 0 & c_{nn} & 0 & d_{nn} & \end{array} \right| \\ & = (a_{11}d_{11} - c_{11}b_{11}) \left| \begin{array}{cc|cc} a_{22} & A'' & b_{22} & B'' \\ & \ddots & & \ddots & \\ 0 & a_{nn} & 0 & b_{nn} & \\ \hline c_{22} & C'' & d_{22} & D'' \\ & \ddots & & \ddots & \\ 0 & c_{nn} & 0 & d_{nn} & \end{array} \right|. \end{aligned}$$

By induction this is equal to

$$(a_{11}d_{11} - c_{11}b_{11}) \prod_{i=2}^n (a_{ii}d_{ii} - b_{ii}c_{ii}) = \prod_{i=1}^n (a_{ii}d_{ii} - b_{ii}c_{ii}).$$

QED

Exercise 5: Let F be a field, n a positive integer, and let V be the space of $n \times n$ matrices over F . If A is a fixed $n \times n$ matrix over F , let T_A be the linear operator on V defined by $T_A(B) = AB - BA$. Consider the family of linear operators T_A obtained by letting A vary over all diagonal matrices. Prove that the operators in that family are simultaneously diagonalizable.

Solution: If we stack the columns of an $n \times n$ matrix on top of each other with column one at the top, the matrix of T_A in the

standard basis is then given by $\begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}$. Thus if A is diagonal then T_A is diagonalizable.

Now $T_A T_B(C) = ABC - ACB - BCA + CBA$ and $T_B T_A(C) = BAC - BCA - ACB + CAB$. Therefore we must show that $BAC + CAB = ABC + CBA$. The i, j -th entry of $BAC + CAB$ is $c_{ij}(a_{ii}b_{ii} + a_{jj}b_{jj})$. And this is exactly the same as the i, j -th entry of $ABC + CBA$. Thus T_A and T_B commute. Thus by Theorem 8 the family can be simultaneously diagonalized.

Section 6.6: Direct-Sum Decompositions

Exercise 1: Let V be a finite-dimensional vector space and let W_1 be any subspace of V . Prove that there is a subspace W_2 of V such that $V = W_1 \oplus W_2$.

Solution:

Exercise 2: Let V be a finite-dimensional vector space and let W_1, \dots, W_k be subspaces of V such that

$$V = W_1 + \dots + W_k \quad \text{and} \quad \dim(V) = \dim(W_1) + \dots + \dim(W_k).$$

Prove that $V = W_1 \oplus \dots \oplus W_k$.

Solution:

Exercise 3: Find a projection E which projects \mathbb{R}^2 onto the subspace spanned by $(1, -1)$ along the subspace spanned by $(1, 2)$.

Solution:

Exercise 4: If E_1 and E_2 are projections onto independent subspaces, then $E_1 + E_2$ is a projection. True or false?

Solution:

Exercise 5: If E is a projection and f is a polynomial, then $f(E) = aI + bE$. What are a and b in terms of the coefficients of f ?

Solution:

Exercise 6: True or false? If a diagonalizable operator has only the characteristic values 0 and 1, it is a projection.

Solution:

Exercise 7: Prove that if E is the projection on R along N , then $(I - E)$ is the projection on N along R .

Solution:

Exercise 8: Let E_1, \dots, E_k be linear operators on the space V such that $E_1 + \dots + E_k = I$.

(a) Prove that if $E_i E_j = 0$ for $i \neq j$, then $E_i^2 = E_i$ for each i .

(b) In the case $k = 2$, prove the converse of (a). That is, if $E_1 + E_2 = I$ and $E_1^2 = E_1, E_2^2 = E_2$, then $E_1 E_2 = 0$.

Solution:

Exercise 9: Let V be a real vector space and E an idempotent linear operator on V , i.e., a projection. Prove that $(I + E)$ is invertible. Find $(I + E)^{-1}$.

Solution:

Exercise 10: Let F be a subfield of the complex numbers (or, a field of characteristic zero). Let V be a finite-dimensional vector space over F . Suppose that E_1, \dots, E_k are projections of V and that $E_1 + \dots + E_k = I$. Prove that $E_i E_j = 0$ for $i \neq j$ (*Hint:* use the trace function and ask yourself what the trace of a projection is.)

Solution:

Project 11: Let V be a vector space, let W_1, \dots, W_k be subspace of V , and let

$$V_j = W_1 + \dots + W_{j-1} + W_{j+1} + \dots + W_k.$$

Suppose that $V = W_1 \oplus \dots \oplus W_k$. Prove that the dual space V^* has the direct-sum decomposition $V^* = V_1^0 \oplus \dots \oplus V_k^0$.

Solution: